

## INTEGRAL FORM OF THE SCHRÖDINGER EQUATION: GROUND STATE OF HYDROGEN

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.9.

We can check that the ground state of the hydrogen atom satisfies the integral form of the Schrödinger equation:

$$(0.1) \quad \psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} V(\mathbf{r}_0) \psi(\mathbf{r}_0) d^3\mathbf{r}_0$$

For the ground state of hydrogen

$$(0.2) \quad \psi_{100}(\mathbf{r}) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

$$(0.3) \quad a \equiv \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

$$(0.4) \quad V(r) = -\frac{e^2}{4\pi\epsilon_0 r} = -\frac{\hbar^2}{mar}$$

$$(0.5) \quad E_1 = -\frac{\hbar^2}{2ma^2}$$

$$(0.6) \quad k = \frac{\sqrt{2mE_1}}{\hbar} = \frac{i}{a}$$

In this case, we're not considering scattering, so the incident plane wave (the free particle) is not present, so  $\psi_0 = 0$  in 0.1, and the integral equation becomes

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(0.7)

$$\psi(\mathbf{r}) = -\frac{m}{2\pi\hbar^2} \left( -\frac{\hbar^2}{ma} \right) \frac{1}{\sqrt{\pi a^3}} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{e^{-|\mathbf{r}-\mathbf{r}_0|/a} e^{-r_0/a}}{|\mathbf{r}-\mathbf{r}_0| r_0} \sin\theta r_0^2 d\phi d\theta dr_0$$

(0.8)

$$= \frac{1}{2\pi^{3/2} a^{5/2}} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{e^{-|\mathbf{r}-\mathbf{r}_0|/a} e^{-r_0/a}}{|\mathbf{r}-\mathbf{r}_0|} \sin\theta r_0 d\phi d\theta dr_0$$

(0.9)

$$= \frac{1}{\sqrt{\pi a^5}} \int_0^\infty \int_0^\pi \frac{e^{-|\mathbf{r}-\mathbf{r}_0|/a} e^{-r_0/a}}{|\mathbf{r}-\mathbf{r}_0|} \sin\theta r_0 d\theta dr_0$$

where we've taken the  $z$  axis to be parallel to  $\mathbf{r}$ , since for the purposes of the integral,  $\mathbf{r}$  is constant.

We have

$$(0.10) \quad |\mathbf{r}-\mathbf{r}_0| = \sqrt{r^2 + r_0^2 - 2rr_0 \cos\theta}$$

so the integral becomes

(0.11)

$$\frac{1}{\sqrt{\pi a^5}} \int_0^\infty \int_0^\pi \frac{e^{-\sqrt{r^2+r_0^2-2rr_0 \cos\theta}/a} e^{-r_0/a}}{\sqrt{r^2+r_0^2-2rr_0 \cos\theta}} \sin\theta r_0 d\theta dr_0 = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

where we did the integral using Maple. This is just the original wave function 0.2 so the integral equation works out.

If you want to do the integral by hand, we do the  $\theta$  integral first since, despite its appearance, it's actually quite simple:

(0.12)

$$\int_0^\pi \frac{e^{-\sqrt{r^2+r_0^2-2rr_0 \cos\theta}/a} e^{-r_0/a}}{\sqrt{r^2+r_0^2-2rr_0 \cos\theta}} \sin\theta r_0 d\theta = -\frac{ae^{-r_0/a}}{r} e^{-\sqrt{r^2+r_0^2-2rr_0 \cos\theta}/a} \Big|_0^\pi$$

The value of the integral depends on whether  $r < r_0$  or  $r > r_0$ :

(0.13)

$$-\frac{ae^{-r_0/a}}{r} e^{-\sqrt{r^2+r_0^2-2rr_0 \cos\theta}/a} \Big|_0^\pi = \begin{cases} \frac{a}{r} \left( e^{2r/a} - 1 \right) e^{-(2r_0+r)/a} & r < r_0 \\ \frac{a}{r} \left( e^{2r_0/a} - 1 \right) e^{-(2r_0+r)/a} & r > r_0 \end{cases}$$

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Using these results, we can split the integral over  $r_0$  into two parts (0 to  $r$  and  $r$  to  $\infty$ ). It is just a simple integral over exponential functions so the answer comes out fairly easily.

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