INTEGRAL FORM OF THE SCHRÖDINGER EQUATION: GROUND STATE OF HYDROGEN

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.9.

We can check that the ground state of the hydrogen atom satisfies the integral form of the Schrödinger equation:

$$\psi(\mathbf{r})=\psi_0(\mathbf{r})-\frac{m}{2\pi\hbar^2}\int\frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}V(\mathbf{r}_0)\,\psi(\mathbf{r}_0)\,d^3\mathbf{r}_0 \tag{1}$$

For the ground state of hydrogen

$$\psi_{100}(\mathbf{r}) = \frac{1}{\sqrt{\pi a^3}}e^{-r/a} \tag{2}$$

$$a \equiv \frac{4\pi\varepsilon_0\hbar^2}{me^2} \tag{3}$$

$$V(r) = -\frac{e^2}{4\pi\varepsilon_0 r}=-\frac{\hbar^2}{mar} \tag{4}$$

$$E_1 = -\frac{\hbar^2}{2ma^2} \tag{5}$$

$$k = \frac{\sqrt{2mE_1}}{\hbar}=\frac{i}{a} \tag{6}$$

In this case, we're not considering scattering, so the incident plane wave (the free particle) is not present, so $\psi_0=0$ in 1, and the integral equation becomes

(7)
$$\psi(\mathbf{r}) = -\frac{m}{2\pi\hbar^2}\left(-\frac{\hbar^2}{ma}\right)\frac{1}{\sqrt{\pi a^3}}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{e^{-|\mathbf{r}-\mathbf{r}_0|/a}e^{-r_0/a}}{|\mathbf{r}-\mathbf{r}_0|\,r_0}\sin\theta r_0^2 d\phi d\theta dr_0$$
(8)
$$= \frac{1}{2\pi^{3/2}a^{5/2}}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{e^{-|\mathbf{r}-\mathbf{r}_0|/a}e^{-r_0/a}}{|\mathbf{r}-\mathbf{r}_0|}\sin\theta r_0 d\phi d\theta dr_0$$
(9)
$$= \frac{1}{\sqrt{\pi a^5}}\int_0^\infty\int_0^\pi\frac{e^{-|\mathbf{r}-\mathbf{r}_0|/a}e^{-r_0/a}}{|\mathbf{r}-\mathbf{r}_0|}\sin\theta r_0 d\theta dr_0$$

where we've taken the z axis to be parallel to $\mathbf{r}$, since for the purposes of the integral, $\mathbf{r}$ is constant.

We have

$$|\mathbf{r}-\mathbf{r}_0| = \sqrt{r^2+r_0^2-2rr_0\cos\theta} \tag{10}$$

so the integral becomes

(11)
$$\frac{1}{\sqrt{\pi a^5}}\int_0^\infty\int_0^\pi\frac{e^{-\sqrt{r^2+r_0^2-2rr_0\cos\theta}/a}e^{-r_0/a}}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}}\sin\theta r_0 d\theta dr_0 = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}$$

where we did the integral using Maple. This is just the original wave function 2 so the integral equation works out.

If you want to do the integral by hand, we do the θ integral first since, despite its appearance, it's actually quite simple:

(12)
$$\int_0^\pi\frac{e^{-\sqrt{r^2+r_0^2-2rr_0\cos\theta}/a}e^{-r_0/a}}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}}\sin\theta r_0 d\theta = -\frac{ae^{-r_0/a}}{r}\,e^{-\sqrt{r^2+r_0^2-2rr_0\cos\theta}/a}\Big|_0^\pi$$

The value of the integral depends on whether $r<r_0$ or $r>r_0$:

(13)
$$-\frac{ae^{-r_0/a}}{r}\,e^{-\sqrt{r^2+r_0^2-2rr_0\cos\theta}/a}\Big|_0^\pi = \begin{cases}\frac{a}{r}\left(e^{2r/a}-1\right)e^{-(2r_0+r)/a} & r<r_0\\ \frac{a}{r}\left(e^{2r_0/a}-1\right)e^{-(2r_0+r)/a} & r>r_0\end{cases}$$

Using these results, we can split the integral over r_0 into two parts (0 to r and r to ∞). It is just a simple integral over exponential functions so the answer comes out fairly easily.

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