

INTEGRAL FORM OF THE SCHRÖDINGER EQUATION: GROUND STATE OF HYDROGEN

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.9.

We can check that the ground state of the hydrogen atom satisfies the integral form of the Schrödinger equation:

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} V(\mathbf{r}_0) \psi(\mathbf{r}_0) d^3\mathbf{r}_0 \quad (1)$$

For the ground state of hydrogen

$$\psi_{100}(\mathbf{r}) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \quad (2)$$

$$a \equiv \frac{4\pi\epsilon_0\hbar^2}{me^2} \quad (3)$$

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r} = -\frac{\hbar^2}{ma r} \quad (4)$$

$$E_1 = -\frac{\hbar^2}{2ma^2} \quad (5)$$

$$k = \frac{\sqrt{2mE_1}}{\hbar} = \frac{i}{a} \quad (6)$$

In this case, we're not considering scattering, so the incident plane wave (the free particle) is not present, so $\psi_0 = 0$ in 1, and the integral equation becomes

$$\psi(\mathbf{r}) = -\frac{m}{2\pi\hbar^2} \left(-\frac{\hbar^2}{ma} \right) \frac{1}{\sqrt{\pi a^3}} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{e^{-|\mathbf{r}-\mathbf{r}_0|/a} e^{-r_0/a}}{|\mathbf{r}-\mathbf{r}_0| r_0} \sin\theta r_0^2 d\phi d\theta dr_0 \quad (7)$$

$$= \frac{1}{2\pi^{3/2} a^{5/2}} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{e^{-|\mathbf{r}-\mathbf{r}_0|/a} e^{-r_0/a}}{|\mathbf{r}-\mathbf{r}_0|} \sin\theta r_0 d\phi d\theta dr_0 \quad (8)$$

$$= \frac{1}{\sqrt{\pi a^5}} \int_0^\infty \int_0^\pi \frac{e^{-|\mathbf{r}-\mathbf{r}_0|/a} e^{-r_0/a}}{|\mathbf{r}-\mathbf{r}_0|} \sin\theta r_0 d\theta dr_0 \quad (9)$$

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where we've taken the z axis to be parallel to \mathbf{r} , since for the purposes of the integral, \mathbf{r} is constant.

We have

$$|\mathbf{r} - \mathbf{r}_0| = \sqrt{r^2 + r_0^2 - 2rr_0 \cos \theta} \quad (10)$$

so the integral becomes

$$\frac{1}{\sqrt{\pi a^5}} \int_0^\infty \int_0^\pi \frac{e^{-\sqrt{r^2 + r_0^2 - 2rr_0 \cos \theta}/a} e^{-r_0/a}}{\sqrt{r^2 + r_0^2 - 2rr_0 \cos \theta}} \sin \theta r_0 d\theta dr_0 = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \quad (11)$$

where we did the integral using Maple. This is just the original wave function 2 so the integral equation works out.

If you want to do the integral by hand, we do the θ integral first since, despite its appearance, it's actually quite simple:

$$\int_0^\pi \frac{e^{-\sqrt{r^2 + r_0^2 - 2rr_0 \cos \theta}/a} e^{-r_0/a}}{\sqrt{r^2 + r_0^2 - 2rr_0 \cos \theta}} \sin \theta r_0 d\theta = -\frac{ae^{-r_0/a}}{r} e^{-\sqrt{r^2 + r_0^2 - 2rr_0 \cos \theta}/a} \Big|_0^\pi \quad (12)$$

The value of the integral depends on whether $r < r_0$ or $r > r_0$:

$$-\frac{ae^{-r_0/a}}{r} e^{-\sqrt{r^2 + r_0^2 - 2rr_0 \cos \theta}/a} \Big|_0^\pi = \begin{cases} \frac{a}{r} (e^{2r/a} - 1) e^{-(2r_0+r)/a} & r < r_0 \\ \frac{a}{r} (e^{2r_0/a} - 1) e^{-(2r_0+r)/a} & r > r_0 \end{cases} \quad (13)$$

Using these results, we can split the integral over r_0 into two parts (0 to r and r to ∞). It is just a simple integral over exponential functions so the answer comes out fairly easily.

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