

FIRST BORN APPROXIMATION: SOFT-SPHERE SCATTERING

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.10.

The first Born approximation for the scattering amplitude comes from the integral form of the Schrödinger equation:

$$(1) \quad \psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} V(\mathbf{r}_0) \psi(\mathbf{r}_0) d^3\mathbf{r}_0$$

This equation is valid for all \mathbf{r} , even for positions near to the origin where the scattering potential V could be significantly different from zero. In a scattering problem, the detector is usually situated far from the scattering region, so for all \mathbf{r} of interest, $r \gg r_0$ and we're well outside the region where $V \neq 0$. In such cases, we can approximate (see Griffiths, section 11.4.2 for details) the integral equation by

$$(2) \quad \psi(\mathbf{r}) \cong Ae^{ikz} - \frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r} \int e^{-ik\cdot\mathbf{r}_0} V(\mathbf{r}_0) \psi(\mathbf{r}_0) d^3\mathbf{r}_0$$

$$(3) \quad = A \left[e^{ikz} - \frac{m}{2\pi\hbar^2 A} \frac{e^{ikr}}{r} \int e^{-ik\cdot\mathbf{r}_0} V(\mathbf{r}_0) \psi(\mathbf{r}_0) d^3\mathbf{r}_0 \right]$$

where $\mathbf{k} \equiv k\hat{\mathbf{r}}$ is a vector pointing from the origin to the detector (that is, parallel to \mathbf{r}). The first term on the RHS represents the incoming plane wave, as usual.

Since the scattering amplitude f is the coefficient of e^{ikr}/r inside the square brackets, we have

$$(4) \quad f(\theta, \phi) = -\frac{m}{2\pi\hbar^2 A} \int e^{-ik\cdot\mathbf{r}_0} V(\mathbf{r}_0) \psi(\mathbf{r}_0) d^3\mathbf{r}_0$$

This formula still doesn't help us much, since we still need to know the wave function ψ inside the scattering region where $V \neq 0$. The Born approximation assumes that the potential is weak, so that the incoming plane wave Ae^{ikz} doesn't change much after it scatters. That is, we assume that, for all points where the integrand is non-zero:

$$(5) \quad \psi(\mathbf{r}_0) \approx \psi_0(\mathbf{r}_0)$$

The incident plane wave has a wave vector of magnitude k that is parallel to $\hat{\mathbf{z}}$, which we can write as

$$(6) \quad \mathbf{k}' \equiv k\hat{\mathbf{z}}$$

For some position \mathbf{r}_0 with z component z_0 :

$$(7) \quad kz_0 = \mathbf{k}' \cdot \mathbf{r}_0$$

so the assumption above amounts to saying that

$$(8) \quad \psi(\mathbf{r}_0) \approx Ae^{i\mathbf{k}' \cdot \mathbf{r}_0}$$

This assumption gives us an approximation for f :

$$(9) \quad f(\theta, \phi) \approx -\frac{m}{2\pi\hbar^2} \int e^{i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}_0} V(\mathbf{r}_0) d^3\mathbf{r}_0$$

It's important to remember that, from the point of view of the integral, \mathbf{k} and \mathbf{k}' are constant, with the direction of $\mathbf{k} = k\hat{\mathbf{r}}$ being how the polar angles θ and ϕ are specified. The vector $\mathbf{k}' = k\hat{\mathbf{z}}$ is always the same as it specifies the direction of the incident particle.

For a spherically symmetric potential, the integral can be simplified a bit by defining the vector

$$(10) \quad \boldsymbol{\kappa} \equiv \mathbf{k}' - \mathbf{k}$$

The vector is the base of an isosceles triangle with sides \mathbf{k}' and \mathbf{k} , so since the angle between \mathbf{k}' and \mathbf{k} is θ (\mathbf{k}' is the direction to the detector and \mathbf{k} is the direction of the incident particle, so θ is the scattering angle), we can divide the isosceles triangle into two symmetric right angled triangles by drawing a line from the origin to the midpoint of $\boldsymbol{\kappa}$. The length of the base is $\kappa/2$ which is also $k \sin \frac{\theta}{2}$, so

$$(11) \quad \kappa = 2k \sin \frac{\theta}{2}$$

Letting the polar axis in the integral 9 lie along $\boldsymbol{\kappa}$ we get $(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}_0 = \kappa r_0 \cos \theta_0$ [where θ_0 is the polar angle of integration, *not* θ !] and

$$(12) \quad f(\theta) \approx -\frac{m}{2\pi\hbar^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{i\kappa r_0 \cos \theta_0} V(r_0) r_0^2 \sin \theta_0 d\phi_0 d\theta_0 dr_0$$

$$(13) \quad = -\frac{2m}{\hbar^2 \kappa} \int_0^\infty V(r_0) r_0 \sin(\kappa r_0) dr_0$$

Example. Soft-sphere scattering. A *soft sphere* is defined by the potential

$$(14) \quad V(\mathbf{r}) = \begin{cases} V_0 & r \leq a \\ 0 & r > a \end{cases}$$

where $V_0 > 0$ is a constant. [The hard sphere takes $V_0 = \infty$.] From 13, we can get the Born approximation for the scattering amplitude:

$$(15) \quad f(\theta) \approx -\frac{2mV_0}{\hbar^2 \kappa} \int_0^a r \sin(\kappa r) dr$$

$$(16) \quad = -\frac{2mV_0}{\hbar^2 \kappa^3} [\sin(\kappa a) - a\kappa \cos(\kappa a)]$$

with the θ dependence given by the definition of κ in 11.

For low energy scattering $\kappa a \ll 1$ and we can expand the sin and cos.

$$(17) \quad \sin(\kappa a) - a\kappa \cos(\kappa a) = \kappa a - \frac{(\kappa a)^3}{3!} + \dots - \kappa a \left(1 - \frac{(\kappa a)^2}{2!} + \dots \right)$$

$$(18) \quad = \frac{(\kappa a)^3}{3} + \dots$$

To this order, the scattering amplitude is

$$(19) \quad f(\theta) \approx -\frac{2mV_0 a^3}{3\hbar^3}$$

which agrees with equation 11.82 in Griffiths.

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