

## FIRST BORN APPROXIMATION: SOFT-SPHERE SCATTERING

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.10.

The first Born approximation for the scattering amplitude comes from the integral form of the Schrödinger equation:

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} V(\mathbf{r}_0) \psi(\mathbf{r}_0) d^3\mathbf{r}_0 \quad (1)$$

This equation is valid for all  $\mathbf{r}$ , even for positions near to the origin where the scattering potential  $V$  could be significantly different from zero. In a scattering problem, the detector is usually situated far from the scattering region, so for all  $\mathbf{r}$  of interest,  $r \gg r_0$  and we're well outside the region where  $V \neq 0$ . In such cases, we can approximate (see Griffiths, section 11.4.2 for details) the integral equation by

$$\psi(\mathbf{r}) \cong Ae^{ikz} - \frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r} \int e^{-i\mathbf{k}\cdot\mathbf{r}_0} V(\mathbf{r}_0) \psi(\mathbf{r}_0) d^3\mathbf{r}_0 \quad (2)$$

$$= A \left[ e^{ikz} - \frac{m}{2\pi\hbar^2 A} \frac{e^{ikr}}{r} \int e^{-i\mathbf{k}\cdot\mathbf{r}_0} V(\mathbf{r}_0) \psi(\mathbf{r}_0) d^3\mathbf{r}_0 \right] \quad (3)$$

where  $\mathbf{k} \equiv k\hat{\mathbf{r}}$  is a vector pointing from the origin to the detector (that is, parallel to  $\mathbf{r}$ ). The first term on the RHS represents the incoming plane wave, as usual.

Since the scattering amplitude  $f$  is the coefficient of  $e^{ikr}/r$  inside the square brackets, we have

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2 A} \int e^{-i\mathbf{k}\cdot\mathbf{r}_0} V(\mathbf{r}_0) \psi(\mathbf{r}_0) d^3\mathbf{r}_0 \quad (4)$$

This formula still doesn't help us much, since we still need to know the wave function  $\psi$  inside the scattering region where  $V \neq 0$ . The Born approximation assumes that the potential is weak, so that the incoming plane wave  $Ae^{ikz}$  doesn't change much after it scatters. That is, we assume that, for all points where the integrand is non-zero:

$$\psi(\mathbf{r}_0) \approx \psi_0(\mathbf{r}_0) \quad (5)$$

The incident plane wave has a wave vector of magnitude  $k$  that is parallel to  $\hat{\mathbf{z}}$ , which we can write as

$$\mathbf{k}' \equiv k\hat{\mathbf{z}} \quad (6)$$

For some position  $\mathbf{r}_0$  with  $z$  component  $z_0$ :

$$kz_0 = \mathbf{k}' \cdot \mathbf{r}_0 \quad (7)$$

so the assumption above amounts to saying that

$$\psi(\mathbf{r}_0) \approx Ae^{i\mathbf{k}' \cdot \mathbf{r}_0} \quad (8)$$

This assumption gives us an approximation for  $f$ :

$$f(\theta, \phi) \approx -\frac{m}{2\pi\hbar^2} \int e^{i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}_0} V(\mathbf{r}_0) d^3\mathbf{r}_0 \quad (9)$$

It's important to remember that, from the point of view of the integral,  $\mathbf{k}$  and  $\mathbf{k}'$  are constant, with the direction of  $\mathbf{k} = k\hat{\mathbf{r}}$  being how the polar angles  $\theta$  and  $\phi$  are specified. The vector  $\mathbf{k}' = k\hat{\mathbf{z}}$  is always the same as it specifies the direction of the incident particle.

For a spherically symmetric potential, the integral can be simplified a bit by defining the vector

$$\kappa \equiv \mathbf{k}' - \mathbf{k} \quad (10)$$

The vector is the base of an isosceles triangle with sides  $\mathbf{k}'$  and  $\mathbf{k}$ , so since the angle between  $\mathbf{k}'$  and  $\mathbf{k}$  is  $\theta$  ( $\mathbf{k}'$  is the direction to the detector and  $\mathbf{k}$  is the direction of the incident particle, so  $\theta$  is the scattering angle), we can divide the isosceles triangle into two symmetric right angled triangles by drawing a line from the origin to the the midpoint of  $\kappa$ . The length of the base is  $\kappa/2$  which is also  $k \sin \frac{\theta}{2}$ , so

$$\kappa = 2k \sin \frac{\theta}{2} \quad (11)$$

Letting the polar axis in the integral 9 lie along  $\kappa$  we get  $(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}_0 = \kappa r_0 \cos \theta_0$  [where  $\theta_0$  is the polar angle of integration, *not*  $\theta$ !] and

$$f(\theta) \approx -\frac{m}{2\pi\hbar^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{i\kappa r_0 \cos \theta_0} V(r_0) r_0^2 \sin \theta_0 d\phi_0 d\theta_0 dr_0 \quad (12)$$

$$= -\frac{2m}{\hbar^2 \kappa} \int_0^\infty V(r_0) r_0 \sin(\kappa r_0) dr_0 \quad (13)$$

**Example.** Soft-sphere scattering. A *soft sphere* is defined by the potential

$$V(\mathbf{r}) = \begin{cases} V_0 & r \leq a \\ 0 & r > a \end{cases} \quad (14)$$

where  $V_0 > 0$  is a constant. [The hard sphere takes  $V_0 = \infty$ .] From 13, we can get the Born approximation for the scattering amplitude:

$$f(\theta) \approx -\frac{2mV_0}{\hbar^2\kappa} \int_0^a r \sin(\kappa r) dr \quad (15)$$

$$= -\frac{2mV_0}{\hbar^2\kappa^3} [\sin(\kappa a) - a\kappa \cos(\kappa a)] \quad (16)$$

with the  $\theta$  dependence given by the definition of  $\kappa$  in 11.

For low energy scattering  $\kappa a \ll 1$  and we can expand the sin and cos.

$$\sin(\kappa a) - a\kappa \cos(\kappa a) = \kappa a - \frac{(\kappa a)^3}{3!} + \dots - \kappa a \left( 1 - \frac{(\kappa a)^2}{2!} + \dots \right) \quad (17)$$

$$= \frac{(\kappa a)^3}{3} + \dots \quad (18)$$

To this order, the scattering amplitude is

$$f(\theta) \approx -\frac{2mV_0 a^3}{3\hbar^3} \quad (19)$$

which agrees with equation 11.82 in Griffiths.

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