

SCATTERING FROM THE YUKAWA POTENTIAL

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problems 11.11-11.12.

We've looked at the Yukawa potential as an example of the variational principle, so here we'll look at scattering by a Yukawa potential, using the first Born approximation. The Yukawa potential in its general form is

$$V(r) = \beta \frac{e^{-\mu r}}{r} \quad (1)$$

where β and μ are constants. Since the potential is spherically symmetric, we can use the Born approximation in the form

$$f(\theta) \approx -\frac{2m}{\hbar^2 \kappa} \int_0^\infty V(r_0) r \sin(\kappa r) dr \quad (2)$$

where

$$\kappa = 2k \sin \frac{\theta}{2} \quad (3)$$

$$k = \frac{\sqrt{2mE}}{\hbar} \quad (4)$$

We get

$$f(\theta) \approx -\frac{2m\beta}{\hbar^2 \kappa} \int_0^\infty e^{-\mu r} \sin(\kappa r) dr \quad (5)$$

$$= -\frac{2m\beta}{\hbar^2 (\kappa^2 + \mu^2)} \quad (6)$$

We did the integral using Maple, but if you want to do it by hand, you can do it with two integrations by parts:

$$\int_0^\infty e^{-\mu r} \sin(\kappa r) dr = -\frac{\cos(\kappa r) e^{-\mu r}}{\kappa} \Big|_0^\infty - \frac{\mu}{\kappa} \int_0^\infty e^{-\mu r} \cos(\kappa r) dr \quad (7)$$

$$= \frac{1}{\kappa} - \frac{\sin(\kappa r) e^{-\mu r}}{\kappa} \Big|_0^\infty - \frac{\mu^2}{\kappa^2} \int_0^\infty e^{-\mu r} \sin(\kappa r) dr \quad (8)$$

$$= \frac{1}{\kappa} - \frac{\mu^2}{\kappa^2} \int_0^\infty e^{-\mu r} \sin(\kappa r) dr \quad (9)$$

$$\left(1 + \frac{\mu^2}{\kappa^2}\right) \int_0^\infty e^{-\mu r} \sin(\kappa r) dr = \frac{1}{\kappa} \quad (10)$$

$$\int_0^\infty e^{-\mu r} \sin(\kappa r) dr = \frac{\kappa}{\mu^2 + \kappa^2} \quad (11)$$

We can find the total cross section by integrating the differential cross section over solid angle:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \quad (12)$$

$$\sigma = \frac{4m^2\beta^2}{\hbar^4} \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{\sin\theta}{(\kappa^2 + \mu^2)^2} \quad (13)$$

$$= \frac{8\pi m^2\beta^2}{\hbar^4} \int_0^\pi d\theta \frac{\sin\theta}{(4k^2 \sin^2 \frac{\theta}{2} + \mu^2)^2} \quad (14)$$

$$= \frac{16\pi m^2\beta^2}{\mu^2 \hbar^4 (\mu^2 + 4k^2)} \quad (15)$$

$$= \frac{16\pi m^2\beta^2}{\mu^2 \hbar^2 (\mu^2 \hbar^2 + 8mE)} \quad (16)$$

Again, we did the integral using Maple. To do it by hand, we use the trig identity

$$\sin\theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \quad (17)$$

followed by the substitution

$$u = \sin \frac{\theta}{2} \quad (18)$$

$$du = \frac{1}{2} \cos \frac{\theta}{2} d\theta \quad (19)$$

This gives

$$\int_0^\pi d\theta \frac{\sin \theta}{(4k^2 \sin^2 \frac{\theta}{2} + \mu^2)^2} = 4 \int_0^1 \frac{u du}{(4k^2 u^2 + \mu^2)^2} \quad (20)$$

$$= -\frac{4}{8k^2 (4k^2 u^2 + \mu^2)} \Big|_0^1 \quad (21)$$

$$= -\frac{1}{2k^2 (4k^2 + \mu^2)} + \frac{1}{2k^2 \mu^2} \quad (22)$$

$$= \frac{2}{(4k^2 + \mu^2) \mu^2} \quad (23)$$