SCATTERING FROM THE YUKAWA POTENTIAL

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problems 11.11-11.12.

We've looked at the Yukawa potential as an example of the variational principle, so here we'll look at scattering by a Yukawa potential, using the first Born approximation. The Yukawa potential in its general form is

$$V(r) = \beta \frac{e^{-\mu r}}{r} \tag{1}$$

where β and μ are constants. Since the potential is spherically symmetric, we can use the Born approximation in the form

$$f(\theta) \approx -\frac{2m}{\hbar^2 \kappa} \int_0^\infty V(r_0)\, r \sin(\kappa r)\, dr \tag{2}$$

where

$$\kappa = 2k \sin\frac{\theta}{2} \tag{3}$$

$$k = \frac{\sqrt{2mE}}{\hbar} \tag{4}$$

We get

$$f(\theta) \approx -\frac{2m\beta}{\hbar^2 \kappa} \int_0^\infty e^{-\mu r} \sin(\kappa r)\, dr \tag{5}$$

$$= -\frac{2m\beta}{\hbar^2 (\kappa^2 + \mu^2)} \tag{6}$$

We did the integral using Maple, but if you want to do it by hand, you can do it with two integrations by parts:

$$\int_0^\infty e^{-\mu r}\sin(\kappa r)\,dr = -\left.\frac{\cos(\kappa r)\,e^{-\mu r}}{\kappa}\right|_0^\infty - \frac{\mu}{\kappa}\int_0^\infty e^{-\mu r}\cos(\kappa r)\,dr \tag{7}$$

$$= \frac{1}{\kappa} - \left.\frac{\sin(\kappa r)\,e^{-\mu r}}{\kappa}\right|_0^\infty - \frac{\mu^2}{\kappa^2}\int_0^\infty e^{-\mu r}\sin(\kappa r)\,dr \tag{8}$$

$$= \frac{1}{\kappa} - \frac{\mu^2}{\kappa^2}\int_0^\infty e^{-\mu r}\sin(\kappa r)\,dr \tag{9}$$

$$\left(1+\frac{\mu^2}{\kappa^2}\right)\int_0^\infty e^{-\mu r}\sin(\kappa r)\,dr = \frac{1}{\kappa} \tag{10}$$

$$\int_0^\infty e^{-\mu r}\sin(\kappa r)\,dr = \frac{\kappa}{\mu^2+\kappa^2} \tag{11}$$

We can find the total cross section by integrating the differential cross section over solid angle:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \tag{12}$$

$$\sigma = \frac{4m^2\beta^2}{\hbar^4}\int_0^\pi d\theta\int_0^{2\pi} d\phi\;\frac{\sin\theta}{(\kappa^2+\mu^2)^2} \tag{13}$$

$$= \frac{8\pi m^2\beta^2}{\hbar^4}\int_0^\pi d\theta\frac{\sin\theta}{\left(4k^2\sin^2\frac{\theta}{2}+\mu^2\right)^2} \tag{14}$$

$$= \frac{16\pi m^2\beta^2}{\mu^2\hbar^4(\mu^2+4k^2)} \tag{15}$$

$$= \frac{16\pi m^2\beta^2}{\mu^2\hbar^2\left(\mu^2\hbar^2+8mE\right)} \tag{16}$$

Again, we did the integral using Maple. To do it by hand, we use the trig identity

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \tag{17}$$

followed by the substitution

$$u = \sin\frac{\theta}{2} \tag{18}$$

$$du = \frac{1}{2}\cos\frac{\theta}{2}\,d\theta \tag{19}$$

This gives

$$\int_0^{\pi} d\theta \frac{\sin\theta}{\left(4k^2\sin^2\frac{\theta}{2}+\mu^2\right)^2} = 4\int_0^1 \frac{u\,du}{\left(4k^2u^2+\mu^2\right)^2} \tag{20}$$

$$= -\frac{4}{8k^2\left(4k^2u^2+\mu^2\right)}\bigg|_0^1 \tag{21}$$

$$= -\frac{1}{2k^2\left(4k^2+\mu^2\right)}+\frac{1}{2k^2\mu^2} \tag{22}$$

$$= \frac{2}{\left(4k^2+\mu^2\right)\mu^2} \tag{23}$$