SCATTERING FROM THE YUKAWA POTENTIAL

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problems 11.11-11.12.

We've looked at the Yukawa potential as an example of the variational principle, so here we'll look at scattering by a Yukawa potential, using the first Born approximation. The Yukawa potential in its general form is

$$V(r)=\beta\frac{e^{-\mu r}}{r} \tag{0.1}$$

where β and μ are constants. Since the potential is spherically symmetric, we can use the Born approximation in the form

$$f(\theta)\approx-\frac{2m}{\hbar^{2}\kappa}\int_{0}^{\infty}V(r_{0})\,r\sin(\kappa r)\,dr \tag{0.2}$$

where

$$\begin{aligned}\kappa &= 2k\sin\frac{\theta}{2} &(0.3)\\ k &= \frac{\sqrt{2mE}}{\hbar} &(0.4)\end{aligned}$$

We get

$$\begin{aligned}f(\theta) &\approx -\frac{2m\beta}{\hbar^{2}\kappa}\int_{0}^{\infty}e^{-\mu r}\sin(\kappa r)\,dr &(0.5)\\ &= -\frac{2m\beta}{\hbar^{2}\left(\kappa^{2}+\mu^{2}\right)} &(0.6)\end{aligned}$$

We did the integral using Maple, but if you want to do it by hand, you can do it with two integrations by parts:

(0.7)
$$\int_0^\infty e^{-\mu r}\sin(\kappa r)\,dr = -\left.\frac{\cos(\kappa r)\,e^{-\mu r}}{\kappa}\right|_0^\infty - \frac{\mu}{\kappa}\int_0^\infty e^{-\mu r}\cos(\kappa r)\,dr$$

(0.8)
$$= \frac{1}{\kappa} - \left.\frac{\sin(\kappa r)\,e^{-\mu r}}{\kappa}\right|_0^\infty - \frac{\mu^2}{\kappa^2}\int_0^\infty e^{-\mu r}\sin(\kappa r)\,dr$$

(0.9)
$$= \frac{1}{\kappa} - \frac{\mu^2}{\kappa^2}\int_0^\infty e^{-\mu r}\sin(\kappa r)\,dr$$

(0.10)
$$\left(1+\frac{\mu^2}{\kappa^2}\right)\int_0^\infty e^{-\mu r}\sin(\kappa r)\,dr = \frac{1}{\kappa}$$

(0.11)
$$\int_0^\infty e^{-\mu r}\sin(\kappa r)\,dr = \frac{\kappa}{\mu^2+\kappa^2}$$

We can find the total cross section by integrating the differential cross section over solid angle:

(0.12)
$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

(0.13)
$$\sigma = \frac{4m^2\beta^2}{\hbar^4}\int_0^\pi d\theta\int_0^{2\pi} d\phi\;\frac{\sin\theta}{\left(\kappa^2+\mu^2\right)^2}$$

(0.14)
$$= \frac{8\pi m^2\beta^2}{\hbar^4}\int_0^\pi d\theta\,\frac{\sin\theta}{\left(4k^2\sin^2\frac{\theta}{2}+\mu^2\right)^2}$$

(0.15)
$$= \frac{16\pi m^2\beta^2}{\mu^2\hbar^4\left(\mu^2+4k^2\right)}$$

(0.16)
$$= \frac{16\pi m^2\beta^2}{\mu^2\hbar^2\left(\mu^2\hbar^2+8mE\right)}$$

Again, we did the integral using Maple. To do it by hand, we use the trig identity

(0.17)
$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

followed by the substitution

$$u = \sin\frac{\theta}{2} \tag{0.18}$$

$$du = \frac{1}{2}\cos\frac{\theta}{2}\,d\theta \tag{0.19}$$

This gives

$$\int_0^{\pi} d\theta \frac{\sin\theta}{\left(4k^2\sin^2\frac{\theta}{2}+\mu^2\right)^2} = 4\int_0^1 \frac{u\,du}{\left(4k^2u^2+\mu^2\right)^2} \tag{0.20}$$

$$= -\frac{4}{8k^2\left(4k^2u^2+\mu^2\right)}\Bigg|_0^1 \tag{0.21}$$

$$= -\frac{1}{2k^2\left(4k^2+\mu^2\right)}+\frac{1}{2k^2\mu^2} \tag{0.22}$$

$$= \frac{2}{\left(4k^2+\mu^2\right)\mu^2} \tag{0.23}$$