BORN APPROXIMATION FOR A SPHERICAL DELTA FUNCTION SHELL

Earlier we looked at scattering from a delta function spherical shell for a low energy incident particle, using partial wave analysis. This was a fairly complex task, as it involved matching interior and exterior wave functions at the delta function boundary.

Here, we’ll calculate the scattering amplitude using the first Born approximation. For a spherically symmetric potential, the approximation is

\[ f(\theta) \approx -\frac{2m}{\hbar^2 \kappa} \int_0^\infty V(r) r \sin(\kappa r) dr \]  

(1)

where

\[ \kappa \equiv 2k \sin \frac{\theta}{2} \]  

(2)

For a delta function potential

\[ V(r) = \alpha \delta(r - a) \]  

(3)

where \( \alpha \) is a constant representing the strength of the delta function, so we get

\[ f(\theta) \approx -\frac{2m\alpha a}{\hbar^2 \kappa} \sin(\kappa a) \]  

(4)

For low energy, \( ka \ll 1 \) so \( \kappa a \ll 1 \) as well, so \( \sin(\kappa a) \approx \kappa a \), and we get

\[ f(\theta) \approx -\frac{2m\alpha a^2}{\hbar^2} \]  

(5)

Our earlier result using partial wave analysis is

\[ f(\theta) \approx -\frac{\beta a}{1 + \beta} \]  

(6)

where
\[ \beta \equiv \frac{2m \alpha a}{\hbar^2} \]  

This gives a differential cross section and total cross section of

\[ \frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \beta^2 a^2 \]  

\[ \sigma = 4\pi \beta^2 a^2 \]  

The low energy result from the Born approximation is, in terms of \( \beta \):

\[ f(\theta) \approx -\beta a \]  

so it agrees with the partial wave result if \( \beta \ll 1 \). This is equivalent to the condition that \( \alpha \ll \hbar^2 / 2ma \), in other words, that the potential is weak. This was the main assumption in deriving the Born approximation, so in this limit, the results are consistent.