IMPULSE APPROXIMATION IN SCATTERING THEORY

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.14.

In classical scattering theory, the simplest approximation is the *impulse approximation*, in which a particle's path is assumed to be a straight line right through the scattering region, and the total impulse resulting from the component of force perpendicular to the particle's trajectory is calculated. This impulse I is assumed to be a small fraction of the particle's incoming horizontal momentum p, so the scattering angle should be small, and given approximately by

$$\theta \approx \arctan \frac{I}{p} \tag{0.1}$$

As an example, we'll apply the impulse approximation to Rutherford scattering of a charge q_1 travelling with kinetic energy E from another charge q_2 at rest. We assume q_1 has an *impact parameter* b (that is, if the particle didn't interact with the target, it would pass by with a closest approach distance of b).

The impulse is the change in momentum that a force produces over a given time, so the required impulse from the perpendicular component of a force is

$$I = \int F_{\perp} dt \tag{0.2}$$

We'll take q_1's straight line trajectory to be along the x axis and place q_2 at $y = b$ on the y axis. Then the Coulomb force between q_1 and q_2 is

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} \tag{0.3}$$

with a perpendicular component of

$$F_{\perp} = F\frac{b}{r} = \frac{q_1 q_2 b}{4\pi\varepsilon_0 r^3} \tag{0.4}$$

The impulse is

$$I=\frac{q_1q_2b}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{dt}{r^3} \tag{0.5}$$

To convert this to an integral over r we can use the fact that q_1's speed is constant at

$$v=\frac{dx}{dt}=\sqrt{\frac{2E}{m}} \tag{0.6}$$

So we have

$$x=\sqrt{r^2-b^2} \tag{0.7}$$

$$dx=\frac{r\,dr}{\sqrt{r^2-b^2}}=\sqrt{\frac{2E}{m}}dt \tag{0.8}$$

$$I=\frac{q_1q_2b}{4\pi\varepsilon_0\sqrt{2E/m}}\times 2\int_b^{\infty}\frac{dr}{r^2\sqrt{r^2-b^2}} \tag{0.9}$$

where the r integral is over the range of r from its closest approach when $r=b$ out to infinity. We've doubled the integral to account for the incoming $(-\infty<x<0)$ and outgoing $(0<x<\infty)$ legs of the journey.

The integral evaluates to

$$\int_b^{\infty}\frac{dr}{r^2\sqrt{r^2-b^2}}=\left.\frac{\sqrt{r^2-b^2}}{b^2r}\right|_b^{\infty} \tag{0.10}$$

$$=\frac{1}{b^2} \tag{0.11}$$

$$I=\frac{q_1q_2}{2\pi\varepsilon_0b\sqrt{2E/m}} \tag{0.12}$$

The incoming momentum is

$$p=m\frac{dx}{dt}=\sqrt{2mE} \tag{0.13}$$

so

$$\theta \approx \arctan\frac{I}{p} \tag{0.14}$$

$$= \arctan\left[\frac{q_1 q_2}{4\pi\varepsilon_0 bE}\right] \tag{0.15}$$

$$b \approx \frac{q_1 q_2}{4\pi\varepsilon_0 E}\cot\theta \tag{0.16}$$

$$\tan\theta \approx \frac{q_1 q_2}{4\pi\varepsilon_0 bE} \tag{0.17}$$

The exact answer is

$$b = \frac{q_1 q_2}{8\pi\varepsilon_0 E}\cot\frac{\theta}{2} \tag{0.18}$$

$$\tan\frac{\theta}{2} = \frac{q_1 q_2}{8\pi\varepsilon_0 bE} \tag{0.19}$$

For small θ, $\tan\frac{\theta}{2} \approx \frac{\theta}{2}$ so

$$\frac{\theta}{2} \approx \frac{q_1 q_2}{8\pi\varepsilon_0 bE} \tag{0.20}$$

$$\theta \approx \frac{q_1 q_2}{4\pi\varepsilon_0 bE} \tag{0.21}$$

which is consistent with 0.17.