SECOND ORDER BORN APPROXIMATION IN SCATTERING THEORY

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.15.

The first order Born approximation in scattering theory is derived from the integral form of the Schrödinger equation

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int g(\mathbf{r}-\mathbf{r}_0) V(\mathbf{r}_0) \psi(\mathbf{r}_0) d^3\mathbf{r}_0 \tag{0.1}$$

where

$$g(\mathbf{r}-\mathbf{r}_0) \equiv -\frac{m}{2\pi\hbar^2} \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} \tag{0.2}$$

is the Green's function and ψ_0 is a free particle wave function. The first order Born approximation assumes that the wave function isn't changed much by the scattering process, so we can approximate the integral equation by replacing the full wave function ψ in the integrand by the incident wave function:

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int g(\mathbf{r}-\mathbf{r}_0) V(\mathbf{r}_0) \psi_0(\mathbf{r}_0) d^3\mathbf{r}_0 \tag{0.3}$$

We can generate a second order approximation by inserting 0.3 into the integral in 0.1. When we do this, it's important to keep track of the position vectors that apply to each integration. We'll relabel $\mathbf{r}_0$ as $\mathbf{r}_1$ in 0.1, and then insert 0.3 into it:

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int g(\mathbf{r}-\mathbf{r}_1) V(\mathbf{r}_1) \psi(\mathbf{r}_1) d^3\mathbf{r}_1 \tag{0.4}$$

$$\approx \psi_0(\mathbf{r}) + \int g(\mathbf{r}-\mathbf{r}_1) V(\mathbf{r}_1) \psi_0(\mathbf{r}_1) d^3\mathbf{r}_1 + \int g(\mathbf{r}-\mathbf{r}_1) V(\mathbf{r}_1) \int g(\mathbf{r}_1-\mathbf{r}_0) V(\mathbf{r}_0) \psi_0(\mathbf{r}_0) d^3\mathbf{r}_0 d^3\mathbf{r}_1 \tag{0.5}$$

The first two terms just repeat the first order Born approximation, so to calculate the second order approximation we need to work out the third term involving the double integral. Using 0.2 we have for this term (which we'll call I_2)

$$I_2=\left(\frac{m}{2\pi\hbar^2}\right)^2\int\int\frac{e^{ik|\mathbf{r}-\mathbf{r}_1|}}{|\mathbf{r}-\mathbf{r}_1|}\frac{e^{ik|\mathbf{r}_1-\mathbf{r}_0|}}{|\mathbf{r}_1-\mathbf{r}_0|}V(\mathbf{r}_1)V(\mathbf{r}_0)\psi_0(\mathbf{r}_0)\,d^3\mathbf{r}_0d^3\mathbf{r}_1 \tag{0.6}$$

In the original derivation, we simplified things by taking $\mathbf{r}$ to point to the detector, which is assumed to be very far from the scattering region. This allows us to approximate the Green's function:

$$g(\mathbf{r}-\mathbf{r}_0)\approx-\frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}e^{-i\mathbf{k}\cdot\mathbf{r}_0} \tag{0.7}$$

This approximation is still valid for the first factor in the integrand in 0.6, but not for the second factor, since $\mathbf{r}_0$ and $\mathbf{r}_1$ are position vectors that both refer to locations within the scattering region (that is, where $V\neq0$). Therefore we can write

$$I_2\approx\left(\frac{m}{2\pi\hbar^2}\right)^2\frac{e^{ikr}}{r}\int\int e^{-i\mathbf{k}\cdot\mathbf{r}_1}\frac{e^{ik|\mathbf{r}_1-\mathbf{r}_0|}}{|\mathbf{r}_1-\mathbf{r}_0|}V(\mathbf{r}_1)V(\mathbf{r}_0)\psi_0(\mathbf{r}_0)\,d^3\mathbf{r}_0d^3\mathbf{r}_1 \tag{0.8}$$

Example. Soft-sphere scattering. A *soft sphere* is defined by the potential

$$V(\mathbf{r})=\begin{cases}V_0 & r\le a\\ 0 & r>a\end{cases} \tag{0.9}$$

where $V_0>0$ is a constant. Griffiths works out the first order Born approximation for the scattering amplitude in his example 11.4, so we'll consider the second order term here. We'll also consider only the case of low energy scattering, which is defined by the condition $ka\ll1$. We start by writing out 0.8 for this case.

$$I_2\approx\left(\frac{m}{2\pi\hbar^2}\right)^2\frac{e^{ikr}}{r}V_0^2\int\int e^{-i\mathbf{k}\cdot\mathbf{r}_1}\frac{e^{ik|\mathbf{r}_1-\mathbf{r}_0|}}{|\mathbf{r}_1-\mathbf{r}_0|}\psi_0(\mathbf{r}_0)\,d^3\mathbf{r}_0d^3\mathbf{r}_1 \tag{0.10}$$

To do the integral, we use an incident plane wave for ψ_0:

$$\psi_0(\mathbf{r}_0) = Ae^{i\mathbf{k}'\cdot\mathbf{r}_0} \tag{0.11}$$

$$\mathbf{k}' \equiv k\hat{\mathbf{z}} \tag{0.12}$$

If we do the $\mathbf{r}_0$ integration first, we can take the polar axis to be along $\mathbf{r}_1$. Then

(0.13)

$$|\mathbf{r}_1 - \mathbf{r}_0| = \sqrt{r_1^2 + r_0^2 - 2r_0 r_1 \cos\theta}$$

(0.14)

$$I_2 \approx \left(\frac{m}{2\pi\hbar^2}\right)^2 \frac{Ae^{ikr}}{r} V_0^2 \int\int e^{-i\mathbf{k}\cdot\mathbf{r}_1} \frac{e^{ik\sqrt{r_1^2+r_0^2-2r_0r_1\cos\theta}}}{\sqrt{r_1^2+r_0^2-2r_0r_1\cos\theta}} e^{i\mathbf{k}'\cdot\mathbf{r}_0} d^3\mathbf{r}_0 d^3\mathbf{r}_1$$

We can't really make much headway with this integral without invoking the low energy assumption. In that case, all the exponentials are approximately 1, so we get

$$I_2 \approx \left(\frac{m}{2\pi\hbar^2}\right)^2 \frac{Ae^{ikr}}{r} V_0^2 \int\int \frac{d^3\mathbf{r}_0 d^3\mathbf{r}_1}{\sqrt{r_1^2+r_0^2-2r_0r_1\cos\theta}} \tag{0.15}$$

Doing the $\mathbf{r}_0$ first, we have

$$\int \frac{d^3\mathbf{r}_0}{\sqrt{r_1^2+r_0^2-2r_0r_1\cos\theta}} = \int_0^a \int_0^\pi \int_0^{2\pi} \frac{r_0^2 \sin\theta\, d\phi d\theta dr_0}{\sqrt{r_1^2+r_0^2-2r_0r_1\cos\theta}} \tag{0.16}$$

$$= 2\pi \int_0^a \int_0^\pi \frac{r_0^2 \sin\theta\, d\theta dr_0}{\sqrt{r_1^2+r_0^2-2r_0r_1\cos\theta}} \tag{0.17}$$

$$= \frac{2\pi}{r_1} \int_0^a r_0 \left[r_0 + r_1 - |r_0 - r_1|\right] dr_0 \tag{0.18}$$

$$= \frac{2\pi}{r_1} \int_0^{r_1} r_0 \left[r_0 + r_1 - (r_1 - r_0)\right] dr_0 + \tag{0.19}$$

$$\frac{2\pi}{r_1} \int_{r_1}^a r_0 \left[r_0 + r_1 - (r_0 - r_1)\right] dr_0$$

$$= \frac{4}{3}\pi r_1^2 + 2\pi a^2 - 2\pi r_1^2 \tag{0.20}$$

$$= 2\pi a^2 - \frac{2}{3}\pi r_1^2 \tag{0.21}$$

The integral over $\mathbf{r}_1$ is now easy, and we get

(0.22)
$$I_2 \approx \left(\frac{m}{2\pi\hbar^2}\right)^2 \frac{Ae^{ikr}}{r} V_0^2 \int_0^a \int_0^\pi \int_0^{2\pi} \left(2\pi a^2 - \frac{2}{3}\pi r_1^2\right) r_1^2 \sin\theta \, d\phi d\theta dr_1$$
(0.23)
$$= \frac{Ae^{ikr}}{r} \frac{8m^2V_0^2a^5}{15\hbar^4}$$

The second order correction to the scattering amplitude is the coefficient of $\frac{Ae^{ikr}}{r}$ so the correction is

$$f_2 = \frac{8m^2V_0^2a^5}{15\hbar^4} \tag{0.24}$$

Combining this with the first order correction taken from Griffiths's Example 11.4 we get

$$f(\theta,\phi) \approx -\frac{2ma^3V_0}{3\hbar^2} + \frac{8m^2V_0^2a^5}{15\hbar^4} \tag{0.25}$$
$$= -\frac{2ma^3V_0}{3\hbar^2}\left(1 - \frac{4mV_0a^2}{5\hbar^2}\right) \tag{0.26}$$