BORN APPROXIMATION OF DELTA FUNCTION WELL AND FINITE SQUARE WELL

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.18.

We can use the one-dimensional Born approximation to get approximate values for transmission through a couple of test potentials. The Born approximation gives a reflection coefficient of

$$R=\left(\frac{m}{\hbar^{2}k}\right)^{2}\left|\int_{-\infty}^{\infty}e^{2ikx}V\left(x\right)dx\right|^{2} \tag{1}$$

with

$$k=\frac{\sqrt{2mE}}{\hbar} \tag{2}$$

Example 1. The delta function well. Here

$$V=-\alpha\delta\left(x\right) \tag{3}$$

so

$$\begin{aligned}R &= \left(\frac{m}{\hbar^{2}k}\right)^{2}\left|-\alpha e^{2ik\times0}\right|^{2} && (4)\\ &= \frac{m\alpha^{2}}{2\hbar^{2}E} && (5)\\ T &= 1-R=1-\frac{m\alpha^{2}}{2\hbar^{2}E} && (6)\end{aligned}$$

The exact values are

$$\begin{aligned}R &= \frac{1}{1+2\hbar^{2}E/m\alpha^{2}} && (7)\\ T &= \frac{1}{1+m\alpha^{2}/2\hbar^{2}E} && (8)\end{aligned}$$

For a weak potential, α is small, so $2\hbar^{2}E/m\alpha^{2}\gg1$ so we can approximate the exact formulas by

$$R \approx \frac{1}{2\hbar^2 E/m\alpha^2} = \frac{m\alpha^2}{2\hbar^2 E} \tag{9}$$

$$T \approx 1 - \frac{m\alpha^2}{2\hbar^2 E} \tag{10}$$

in agreement with the Born approximation.

Example 2. The finite square well. Here

$$V = \begin{cases} -V_0 & -a < x < a \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

We get from 1

$$\int_{-\infty}^{\infty} e^{2ikx} V(x)\, dx = -V_0 \int_{-a}^{a} e^{2ikx} dx \tag{12}$$

$$= -V_0 \frac{e^{2ika} - e^{-2ika}}{2ik} \tag{13}$$

$$= -\frac{V_0}{k} \sin(2ka) \tag{14}$$

$$R = \left(\frac{m}{\hbar^2 k}\right)^2 \left| -\frac{V_0}{k} \sin(2ka) \right|^2 \tag{15}$$

$$= \frac{V_0^2}{4E^2} \sin^2\left(\frac{2a\sqrt{2mE}}{\hbar}\right) \tag{16}$$

$$T = 1 - \frac{V_0^2}{4E^2} \sin^2\left(\frac{2a\sqrt{2mE}}{\hbar}\right) \tag{17}$$

The exact formula is

$$T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right) \tag{18}$$

For a weak potential (or, equivalently, a high incident energy), $E \gg V_0$ and we get

$$T^{-1} \approx 1 + \frac{V_0^2}{4E^2} \sin^2\left(\frac{2a\sqrt{2mE}}{\hbar}\right) \tag{19}$$

$$T \approx 1 - \frac{V_0^2}{4E^2} \sin^2\left(\frac{2a\sqrt{2mE}}{\hbar}\right) \tag{20}$$

so again, the Born approximation gives a good result.