

BORN APPROXIMATION OF DELTA FUNCTION WELL AND FINITE SQUARE WELL

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 11.18.

We can use the one-dimensional Born approximation to get approximate values for transmission through a couple of test potentials. The Born approximation gives a reflection coefficient of

$$(1) \quad R = \left(\frac{m}{\hbar^2 k} \right)^2 \left| \int_{-\infty}^{\infty} e^{2ikx} V(x) dx \right|^2$$

with

$$(2) \quad k = \frac{\sqrt{2mE}}{\hbar}$$

Example 1. The delta function well. Here

$$(3) \quad V = -\alpha \delta(x)$$

so

$$(4) \quad R = \left(\frac{m}{\hbar^2 k} \right)^2 \left| -\alpha e^{2ik \times 0} \right|^2$$

$$(5) \quad = \frac{m\alpha^2}{2\hbar^2 E}$$

$$(6) \quad T = 1 - R = 1 - \frac{m\alpha^2}{2\hbar^2 E}$$

The exact values are

$$(7) \quad R = \frac{1}{1 + 2\hbar^2 E / m\alpha^2}$$

$$(8) \quad T = \frac{1}{1 + m\alpha^2 / 2\hbar^2 E}$$

For a weak potential, α is small, so $2\hbar^2 E/m\alpha^2 \gg 1$ so we can approximate the exact formulas by

$$(9) \quad R \approx \frac{1}{2\hbar^2 E/m\alpha^2} = \frac{m\alpha^2}{2\hbar^2 E}$$

$$(10) \quad T \approx 1 - \frac{m\alpha^2}{2\hbar^2 E}$$

in agreement with the Born approximation.

Example 2. The finite square well. Here

$$(11) \quad V = \begin{cases} -V_0 & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

We get from 1

$$(12) \quad \int_{-\infty}^{\infty} e^{2ikx} V(x) dx = -V_0 \int_{-a}^a e^{2ikx} dx$$

$$(13) \quad = -V_0 \frac{e^{2ika} - e^{-2ika}}{2ik}$$

$$(14) \quad = -\frac{V_0}{k} \sin(2ka)$$

$$(15) \quad R = \left(\frac{m}{\hbar^2 k} \right)^2 \left| -\frac{V_0}{k} \sin(2ka) \right|^2$$

$$(16) \quad = \frac{V_0^2}{4E^2} \sin^2 \left(\frac{2a\sqrt{2mE}}{\hbar} \right)$$

$$(17) \quad T = 1 - \frac{V_0^2}{4E^2} \sin^2 \left(\frac{2a\sqrt{2mE}}{\hbar} \right)$$

The exact formula is

$$(18) \quad T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2 \left(\frac{2a}{\hbar} \sqrt{2m(E+V_0)} \right)$$

For a weak potential (or, equivalently, a high incident energy), $E \gg V_0$ and we get

$$(19) \quad T^{-1} \approx 1 + \frac{V_0^2}{4E^2} \sin^2 \left(\frac{2a\sqrt{2mE}}{\hbar} \right)$$

$$(20) \quad T \approx 1 - \frac{V_0^2}{4E^2} \sin^2 \left(\frac{2a\sqrt{2mE}}{\hbar} \right)$$

so again, the Born approximation gives a good result.