HARMONIC OSCILLATOR: ALGEBRAIC SOLUTION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Sec 2.3.

The quantum harmonic oscillator can be approached either by an algebraic method or by solving the Schrödinger equation directly. We'll examine the algebraic approach here.

We still start with the Schrödinger equation, which for the harmonic oscillator is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}kx^2\psi=E\psi \tag{0.1}$$

where $k=m\omega^2$ and m is the mass and ω is the angular frequency of oscillation. Using the form for the quantum momentum operator, we can write the equation as

$$\frac{1}{2m}\left[p^2+(m\omega x)^2\right]\psi=E\psi \tag{0.2}$$

The sum in the brackets suggests we try to factor it along the lines of

$$a^2+b^2=(ia+b)(-ia+b) \tag{0.3}$$

However, this doesn't work here, since the position and momentum operators don't commute. As we've seen, the commutator is

$$[x,p]=i\hbar \tag{0.4}$$

If we try the factoring by looking at the two operators:

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}\left[-ip+m\omega x\right] \tag{0.5}$$

$$a_- = \frac{1}{\sqrt{2\hbar m\omega}}\left[ip+m\omega x\right] \tag{0.6}$$

then multiplying them together:

$$a_-a_+ = \frac{1}{2\hbar m\omega}\left[ip+m\omega x\right]\left[-ip+m\omega x\right] \tag{0.7}$$

$$= \frac{1}{2\hbar m\omega}\left[p^2+(m\omega x)^2-im\omega\left[x,p\right]\right] \tag{0.8}$$

$$= \frac{1}{2\hbar m\omega}\left[p^2+(m\omega x)^2+m\omega\hbar\right] \tag{0.9}$$

$$= \frac{H}{\hbar\omega}+\frac{1}{2} \tag{0.10}$$

where H is the Hamiltonian from the original equation.

The calculation can be repeated starting with a_+a_- and we get

$$a_+a_- = \frac{H}{\hbar\omega}-\frac{1}{2} \tag{0.11}$$

From these two results we get the commutator for the $a_\pm$ operators:

$$\left[a_-,a_+\right] = 1 \tag{0.12}$$

The original Schrödinger equation can therefore be written as

$$\hbar\omega\left[a_\pm a_\mp \pm \frac{1}{2}\right]\psi = E\psi \tag{0.13}$$

At this point you'd be entitled to say 'so what?' since it's not obvious we've made any progress. However, the operators $a_\pm$ have special properties which we need a bit of algebra to demonstrate.

Consider a_+ first. If ψ satisfies the Schrödinger equation with energy E, then $a_+\psi$ satisfies the same equation but with energy $E+\hbar\omega$. To see this, watch carefully:

$$H(a_+\psi) = \hbar\omega\left[a_+a_- + \frac{1}{2}\right](a_+\psi) \tag{0.14}$$
$$= \hbar\omega\left[a_+a_-a_+ + \frac{1}{2}a_+\right]\psi \tag{0.15}$$
$$= \hbar\omega a_+\left[a_-a_+ + \frac{1}{2}\right]\psi \tag{0.16}$$
$$= \hbar\omega a_+\left[a_+a_- + \frac{1}{2} + 1\right]\psi \tag{0.17}$$
$$= a_+(H+\hbar\omega)\psi \tag{0.18}$$
$$= a_+(E+\hbar\omega)\psi \tag{0.19}$$
$$= (E+\hbar\omega)a_+\psi \tag{0.20}$$

In the fourth line, we used the commutator 0.12 and in the last line we used the fact that any operator commutes with a constant.

By the same argument, we can show that $a_-\psi$ is a wave function with energy $E-\hbar\omega$. For this reason a_+ and a_- are called the *raising* and *lowering* operators, respectively.

Now, since the harmonic oscillator potential is parabolic, we'd expect no upper limit to the energy levels, but we would expect a lower limit, so that applying the lowering operator to the ground state should give us zero. If we call the ground state ψ_0 then we get

$$a_-\psi_0 = 0 \tag{0.21}$$
$$\frac{1}{\sqrt{2\hbar m\omega}}[ip+m\omega x]\psi_0 = 0 \tag{0.22}$$
$$\hbar\frac{d\psi_0}{dx} + m\omega x\psi_0 = 0 \tag{0.23}$$

We can solve this by the usual method of separating the dependent and independent variables:

$$\int\frac{d\psi_0}{\psi_0} = -\frac{m\omega}{\hbar}\int x\,dx \tag{0.24}$$
$$\ln\psi_0 = -\frac{m\omega}{2\hbar}x^2 + C \tag{0.25}$$
$$\psi_0 = Ae^{-m\omega x^2/2\hbar} \tag{0.26}$$

where $A = e^C$ is the constant of integration.

We can normalize this function using the standard integral

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \tag{0.27}$$

so we get

$$\begin{aligned} 1 &= |A|^2 \int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar} dx && (0.28) \\ &= |A|^2 \sqrt{\frac{\hbar\pi}{m\omega}} && (0.29) \\ A &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} && (0.30) \end{aligned}$$

With this as our starting point, we could generate all the higher wave functions by applying the raising operator, although this gets pretty tedious after the first few. One thing is worthy of note however. All we showed above is that applying the raising operator to a stationary state with energy E gives a function with energy $E + \hbar\omega$, but we didn't show that such a function would be normalized, and in fact, in practice, it's not. All we can say at this stage then is that

$$\psi_n = A_n (a_+)^n \psi_0 \tag{0.31}$$

where the normalization constant A_n must be found.

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