

HARMONIC OSCILLATOR: ALGEBRAIC NORMALIZATION OF RAISING AND LOWERING OPERATORS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Sec 2.3.

In the study of the harmonic oscillator, the raising and lowering operators can be used to generate successive stationary states. The operators are:

$$(0.1) \quad a_+ = \frac{1}{\sqrt{2\hbar m\omega}} [-ip + m\omega x]$$

$$(0.2) \quad a_- = \frac{1}{\sqrt{2\hbar m\omega}} [ip + m\omega x]$$

As we saw in the last post, these operators only give the form of the wave function up to a normalization constant A_n which we still need to determine.

$$(0.3) \quad \psi_n = A_n (a_+)^n \psi_0$$

$$(0.4) \quad E_n = \left(n + \frac{1}{2} \right) \hbar\omega$$

We can rewrite the operators in explicit derivative form as

$$(0.5) \quad a_+ = \frac{1}{\sqrt{2\hbar m\omega}} \left[-\hbar \frac{d}{dx} + m\omega x \right]$$

$$(0.6) \quad a_- = \frac{1}{\sqrt{2\hbar m\omega}} \left[\hbar \frac{d}{dx} + m\omega x \right]$$

If we consider the term involving the derivative only, we can use integration by parts to examine the integral:

$$(0.7) \quad \int_{-\infty}^{\infty} \psi_n^*(x) \frac{d}{dx} \psi_m(x) dx = \psi_n^*(x) \psi_m(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dx} [\psi_n^*(x)] \psi_m(x) dx$$

Since the harmonic oscillator stationary states $\psi_n(x)$ arise from a potential that goes to infinity at $x = \pm\infty$, all the wave functions must go to zero at

infinity, so we can discard the first term on the right. We are therefore left with

$$(0.8) \quad \int_{-\infty}^{\infty} \psi_n^*(x) \frac{d}{dx} \psi_m(x) dx = - \int_{-\infty}^{\infty} \frac{d}{dx} [\psi_n^*(x)] \psi_m(x) dx$$

The second term in each operator is a simple multiplier so it doesn't matter where it appears in the integral. That is

$$(0.9) \quad \int_{-\infty}^{\infty} \psi_n^*(x) x \psi_m(x) dx = \int_{-\infty}^{\infty} x \psi_n^*(x) \psi_m(x) dx$$

Using these two relations, we see that

$$(0.10) \quad \int_{-\infty}^{\infty} \psi_n^*(x) a_+ [\psi_m(x)] dx = \int_{-\infty}^{\infty} a_- [\psi_n^*(x)] \psi_m(x) dx$$

$$(0.11) \quad \int_{-\infty}^{\infty} \psi_n^*(x) a_- [\psi_m(x)] dx = \int_{-\infty}^{\infty} a_+ [\psi_n^*(x)] \psi_m(x) dx$$

If we apply a raising operator followed by a lowering operator (or vice versa) we will get back the same function multiplied by a constant. From the last post, the Schrödinger equation for the oscillator can be written as

$$(0.12) \quad \hbar\omega \left[a_{\pm} a_{\mp} \pm \frac{1}{2} \right] \psi_n = \left(n + \frac{1}{2} \right) \hbar\omega \psi_n$$

so we must have

$$(0.13) \quad a_+ a_- \psi_n = n \psi_n$$

$$(0.14) \quad a_- a_+ \psi_n = (n+1) \psi_n$$

Plugging this back into an integral of the form above, we get

$$(0.15) \quad \int_{-\infty}^{\infty} a_+ [\psi_n^*(x)] a_+ [\psi_n(x)] dx = \int_{-\infty}^{\infty} [a_- a_+ \psi_n^*(x)] \psi_n(x) dx$$

$$(0.16) \quad = (n+1) \int_{-\infty}^{\infty} \psi_n^*(x) \psi_n(x) dx$$

$$(0.17) \quad = n+1$$

where the last line is a result of $\psi_n(x)$ being normalized.

Since $a_+ \psi_n$ is a constant times ψ_{n+1} we must have

$$(0.18) \quad a_+ \psi_n = \sqrt{n+1} \psi_{n+1}$$

By a similar argument with the lowering operator, we get

$$(0.19) \quad a_- \psi_n = \sqrt{n} \psi_{n-1}$$

Note in particular that when $n = 0$ the lowering operator does in fact give zero when operating on the ground state.

Thus to get a normalized function, we must have

$$(0.20) \quad \psi_{n+1} = \frac{1}{\sqrt{n+1}} a_+ \psi_n$$

or in general, by recursion

$$(0.21) \quad \psi_n = \frac{1}{\sqrt{n!}} (a_+)^n \psi_0$$

$$(0.22) \quad = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{1}{\sqrt{n!}} (a_+)^n e^{-m\omega x^2/2\hbar}$$

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