HARMONIC OSCILLATOR: ALGEBRAIC NORMALIZATION OF RAISING AND LOWERING OPERATORS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Sec 2.3.

In the study of the harmonic oscillator, the raising and lowering operators can be used to generate successive stationary states. The operators are:

$$a_{+} = \frac{1}{\sqrt{2\hbar m\omega}}\left[-ip + m\omega x\right] \tag{1}$$

$$a_{-} = \frac{1}{\sqrt{2\hbar m\omega}}\left[ip + m\omega x\right] \tag{2}$$

As we saw in the last post, these operators only give the form of the wave function up to a normalization constant A_n which we still need to determine.

$$\psi_n = A_n \left(a_{+}\right)^n \psi_0 \tag{3}$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{4}$$

We can rewrite the operators in explicit derivative form as

$$a_{+} = \frac{1}{\sqrt{2\hbar m\omega}}\left[-\hbar\frac{d}{dx} + m\omega x\right] \tag{5}$$

$$a_{-} = \frac{1}{\sqrt{2\hbar m\omega}}\left[\hbar\frac{d}{dx} + m\omega x\right] \tag{6}$$

If we consider the term involving the derivative only, we can use integration by parts to examine the integral:

$$\int_{-\infty}^{\infty}\psi_n^*(x)\frac{d}{dx}\psi_m(x)dx = \psi_n^*(x)\psi_m(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}\left[\psi_n^*(x)\right]\psi_m(x)dx \tag{7}$$

Since the harmonic oscillator stationary states $\psi_n(x)$ arise from a potential that goes to infinity at $x = \pm\infty$, all the wave functions must go to zero at infinity, so we can discard the first term on the right. We are therefore left with

$$\int_{-\infty}^{\infty}\psi_n^*(x)\frac{d}{dx}\psi_m(x)dx = -\int_{-\infty}^{\infty}\frac{d}{dx}\left[\psi_n^*(x)\right]\psi_m(x)dx \tag{8}$$

The second term in each operator is a simple multiplier so it doesn't matter where it appears in the integral. That is

$$\int_{-\infty}^{\infty}\psi_n^*(x)x\psi_m(x)dx = \int_{-\infty}^{\infty}x\psi_n^*(x)\psi_m(x)dx \tag{9}$$

Using these two relations, we see that

$$\int_{-\infty}^{\infty}\psi_n^*(x)a_+\left[\psi_m(x)\right]dx = \int_{-\infty}^{\infty}a_-\left[\psi_n^*(x)\right]\psi_m(x)dx \tag{10}$$

$$\int_{-\infty}^{\infty}\psi_n^*(x)a_-\left[\psi_m(x)\right]dx = \int_{-\infty}^{\infty}a_+\left[\psi_n^*(x)\right]\psi_m(x)dx \tag{11}$$

If we apply a raising operator followed by a lowering operator (or vice versa) we will get back the same function multiplied by a constant. From the last post, the Schrödinger equation for the oscillator can be written as

$$\hbar\omega\left[a_\pm a_\mp \pm \frac{1}{2}\right]\psi_n = \left(n+\frac{1}{2}\right)\hbar\omega\psi_n \tag{12}$$

so we must have

$$a_+a_-\psi_n = n\psi_n \tag{13}$$

$$a_-a_+\psi_n = (n+1)\psi_n \tag{14}$$

Plugging this back into an integral of the form above, we get

$$\int_{-\infty}^{\infty}a_+\left[\psi_n^*(x)\right]a_+\left[\psi_n(x)\right]dx = \int_{-\infty}^{\infty}\left[a_-a_+\psi_n^*(x)\right]\psi_n(x)dx \tag{15}$$

$$= (n+1)\int_{-\infty}^{\infty}\psi_n^*(x)\psi_n(x)dx \tag{16}$$

$$= n+1 \tag{17}$$

where the last line is a result of $\psi_n(x)$ being normalized.

Since $a_+\psi_n$ is a constant times ψ_{n+1} we must have

$$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1} \tag{18}$$

By a similar argument with the lowering operator, we get

$$a_-\psi_n = \sqrt{n}\psi_{n-1} \tag{19}$$

Note in particular that when $n=0$ the lowering operator does in fact give zero when operating on the ground state.

Thus to get a normalized function, we must have

$$\psi_{n+1} = \frac{1}{\sqrt{n+1}}a_+\psi_n \tag{20}$$

or in general, by recursion

$$\psi_n = \frac{1}{\sqrt{n!}}\left(a_+\right)^n\psi_0 \tag{21}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{n!}}\left(a_+\right)^n e^{-m\omega x^2/2\hbar} \tag{22}$$

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