

HARMONIC OSCILLATOR: ALGEBRAIC NORMALIZATION OF RAISING AND LOWERING OPERATORS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Sec 2.3.

In the study of the harmonic oscillator, the raising and lowering operators can be used to generate successive stationary states. The operators are:

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}} [-ip + m\omega x] \quad (1)$$

$$a_- = \frac{1}{\sqrt{2\hbar m\omega}} [ip + m\omega x] \quad (2)$$

As we saw in the last post, these operators only give the form of the wave function up to a normalization constant A_n which we still need to determine.

$$\psi_n = A_n (a_+)^n \psi_0 \quad (3)$$

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega \quad (4)$$

We can rewrite the operators in explicit derivative form as

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}} \left[-\hbar \frac{d}{dx} + m\omega x \right] \quad (5)$$

$$a_- = \frac{1}{\sqrt{2\hbar m\omega}} \left[\hbar \frac{d}{dx} + m\omega x \right] \quad (6)$$

If we consider the term involving the derivative only, we can use integration by parts to examine the integral:

$$\int_{-\infty}^{\infty} \psi_n^*(x) \frac{d}{dx} \psi_m(x) dx = \psi_n^*(x) \psi_m(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dx} [\psi_n^*(x)] \psi_m(x) dx \quad (7)$$

Since the harmonic oscillator stationary states $\psi_n(x)$ arise from a potential that goes to infinity at $x = \pm\infty$, all the wave functions must go to zero at infinity, so we can discard the first term on the right. We are therefore left with

$$\int_{-\infty}^{\infty} \psi_n^*(x) \frac{d}{dx} \psi_m(x) dx = - \int_{-\infty}^{\infty} \frac{d}{dx} [\psi_n^*(x)] \psi_m(x) dx \quad (8)$$

The second term in each operator is a simple multiplier so it doesn't matter where it appears in the integral. That is

$$\int_{-\infty}^{\infty} \psi_n^*(x) x \psi_m(x) dx = \int_{-\infty}^{\infty} x \psi_n^*(x) \psi_m(x) dx \quad (9)$$

Using these two relations, we see that

$$\int_{-\infty}^{\infty} \psi_n^*(x) a_+ [\psi_m(x)] dx = \int_{-\infty}^{\infty} a_- [\psi_n^*(x)] \psi_m(x) dx \quad (10)$$

$$\int_{-\infty}^{\infty} \psi_n^*(x) a_- [\psi_m(x)] dx = \int_{-\infty}^{\infty} a_+ [\psi_n^*(x)] \psi_m(x) dx \quad (11)$$

If we apply a raising operator followed by a lowering operator (or vice versa) we will get back the same function multiplied by a constant. From the last post, the Schrödinger equation for the oscillator can be written as

$$\hbar\omega \left[a_{\pm} a_{\mp} \pm \frac{1}{2} \right] \psi_n = \left(n + \frac{1}{2} \right) \hbar\omega \psi_n \quad (12)$$

so we must have

$$a_+ a_- \psi_n = n \psi_n \quad (13)$$

$$a_- a_+ \psi_n = (n+1) \psi_n \quad (14)$$

Plugging this back into an integral of the form above, we get

$$\int_{-\infty}^{\infty} a_+ [\psi_n^*(x)] a_+ [\psi_n(x)] dx = \int_{-\infty}^{\infty} [a_- a_+ \psi_n^*(x)] \psi_n(x) dx \quad (15)$$

$$= (n+1) \int_{-\infty}^{\infty} \psi_n^*(x) \psi_n(x) dx \quad (16)$$

$$= n+1 \quad (17)$$

where the last line is a result of $\psi_n(x)$ being normalized.

Since $a_+ \psi_n$ is a constant times ψ_{n+1} we must have

$$a_+ \psi_n = \sqrt{n+1} \psi_{n+1} \quad (18)$$

By a similar argument with the lowering operator, we get

$$a_- \psi_n = \sqrt{n} \psi_{n-1} \quad (19)$$

Note in particular that when $n = 0$ the lowering operator does in fact give zero when operating on the ground state.

Thus to get a normalized function, we must have

$$\psi_{n+1} = \frac{1}{\sqrt{n+1}} a_+ \psi_n \quad (20)$$

or in general, by recursion

$$\psi_n = \frac{1}{\sqrt{n!}} (a_+)^n \psi_0 \quad (21)$$

$$= \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{1}{\sqrt{n!}} (a_+)^n e^{-m\omega x^2/2\hbar} \quad (22)$$

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