

HARMONIC OSCILLATOR - SUMMARY

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Sec 2.3.

In classical physics, the harmonic oscillator is quite simple to study. It is defined by the force law

$$(1) \quad F = -kx$$

where x is the displacement of the particle from the equilibrium position (the point at which no force acts), and k is a positive constant. The force thus acts always in a way that tries to pull the particle back towards the equilibrium position, and k is a measure of how strong the force is.

The simplest example of a classical harmonic oscillator is a mass m attached to a spring. In that case, it's not too hard to show that if the mass is pulled away from its equilibrium position to a point x_0 and released, then it undergoes simple harmonic motion with a frequency of $\omega = \sqrt{k/m}$. That is, the position as a function of time is

$$(2) \quad x(t) = x_0 \cos(\omega t)$$

This force can be derived from a potential function, which is found by integrating the force law:

$$(3) \quad V(x) = -\int_0^x F dx'$$

$$(4) \quad = \frac{1}{2}kx^2$$

Since the force can be derived from a potential, it is a conservative force, which means that the work done by a particle moving in a closed loop under the influence of the force is zero.

The total energy of a harmonic oscillator is either the maximum kinetic or maximum potential energy, both of which are (for the initial conditions specified here) $\frac{1}{2}kx_0^2$, and the energy can take any positive value for a given value of k merely by adjusting x_0 (in the case of the spring, we can adjust the

amount of energy in the system by pulling the mass out to various starting points x_0).

In the quantum mechanical treatment of the harmonic oscillator, the problem is that of solving the Schrödinger equation with the potential $V(x) = \frac{1}{2}kx^2$. Since the potential is time-independent, the Schrödinger equation is separable, so the problem reduces to the solution of the ordinary differential equation:

$$(5) \quad -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi = E\psi$$

Exact solutions of this equation have been found, but the mathematical methods involved are fairly complicated, so it will take several posts to analyze them all. The purpose of this post is to summarize the results, and the most important mathematical techniques required to obtain these results. Other posts will be written in due course describing these methods.

First, the harmonic oscillator potential shares one important feature with that of the particle in a box (infinite square well). Since the potential is a parabola in x , its value increases to infinity in both the positive and negative directions. We would, therefore, expect that only discrete energy levels are allowed, rather than the continuum of energy values possible in classical mechanics. This turns out to be true, and the energy levels of the harmonic oscillator are

$$(6) \quad E_n = \left(n + \frac{1}{2}\right) \hbar\omega$$

where $\omega = \sqrt{k/m}$ as in the classical case. Here, $n = 0, 1, 2, \dots$ so the ground state of the oscillator has a non-zero energy of $E_0 = \hbar\omega/2$. Just as with the particle in a box, there is a zero-point energy in the system, so that the particle cannot have zero energy.

There are two main approaches that are taken to obtain the energy levels and stationary states of the harmonic oscillator. One is the *algebraic method* (not using calculus to any significant degree), in which the energy levels are obtained along with a way of generating all the stationary states starting with the ground state ψ_0 . This method is an elegant way of finding the quantized energies, although it is quite a tedious way of generating the higher level stationary states. The method is worth learning since it is good practice for algebraic methods used in quantum field theory.

The other method involves confronting the Schrödinger equation directly, and finding the solutions ψ_n . This method uses a couple of tools that are frequently found in the solution of differential equations: finding a solution

for the asymptotic form of the equation (that is, a solution that works for very large values of x), then factoring out this solution and trying to solve the differential equation that is left over. This latter differential equation can be solved by proposing a solution as an infinite series, in which a sum of powers of x is plugged into the differential equation and conditions are imposed on the coefficients of the powers of x . This latter method results in solutions that contain *Hermite polynomials*, and provide closed form solutions for the stationary states.

Once the stationary states are found, the general solution, as always for solutions of a separable Schrödinger equation, are linear combinations of the stationary states with the time term restored:

$$(7) \quad \Psi(x, t) = \sum_{n=0}^{\infty} c_n \psi_n e^{-iE_n t / \hbar}$$

Notice that in this case, the sum starts at zero rather than the more usual 1.

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