

## LIMITS

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On the page explaining why physics needs calculus, we saw that in order to calculate the speed of something at a precise place or time, rather than merely its average value over some interval, we need the mathematical concept of a limit. To get used to the idea of a limit, suppose we consider the function

$$(0.1) \quad f(x) = \frac{x^2 - 1}{x - 1}$$

For all values except  $x = 1$  (where the denominator becomes zero) this ratio has a well-defined value which you can work out by doing a bit of arithmetic. But if you calculate the value of  $f(x)$  as  $x$  gets closer and closer to 1 (from either side), you'll find that the value gets closer and closer to 2, although it is never exactly equal to 2.

The algebraically gifted among you may notice that the numerator of  $f(x)$  factors so we can write

$$(0.2) \quad f(x) = \frac{(x+1)(x-1)}{x-1} = x+1$$

and it is therefore “obvious” that  $f(1) = 2$ . But in its original form,  $f(1)$  is undefined, because we are facing the indeterminate ratio of zero divided by zero, so the second equality in the above “derivation” isn't really valid as it stands. Whenever we are faced with a situation like this, where plugging in a value for  $x$  would result in an illegal operation (such as dividing by zero), the only correct way to resolve the situation is by taking a limit.

So what exactly *is* a limit, mathematically speaking? Using the above function as an example, what we would like to prove is that if we restrict  $x$  to be very close to 1, then  $f(x)$  is always restricted to be very close to 2.

Obviously, terms like “very close to” are not mathematically precise, so we need an exact way of stating the requirements in mathematical notation. The standard way of doing this is by means of epsilon-delta proofs, which are so-called simply because they use the Greek letters  $\epsilon$  and  $\delta$ .

If we want to prove that the limit of  $f(x)$  as  $x \rightarrow 1$  is 2, then we need to show the following.

If  $|x - 1| < \delta$ , then  $|f(x) - 2| < \varepsilon$  where  $\varepsilon$  and  $\delta$  are assumed to be very small quantities. It might seem that this doesn't say very much, since at this stage it looks like we can choose  $\varepsilon$  and  $\delta$  independently so we could make both these inequalities true all the time. The trick is that we can (if the limit is what we think it is) show that if  $|x - 1| < \delta$  then the second condition is implied, and that we can find a relation between  $\varepsilon$  and  $\delta$ . In other words, we are saying that if  $x$  is forced to be very close to 1, then  $f(x)$  is forced to be very close to 2. Note that the converse statement (that is, specifying that if  $f(x)$  is very close to some value, then  $x$  must be very close to some other value) is not necessarily true, since a function might approach a particular value for several different values of  $x$ . Having said that, however, sometimes it is easier to find the proof of a limit if we work backwards, and that happens to be true for this function.

So let's see what happens if we assume  $|f(x) - 2| < \varepsilon$

$$(0.3) \quad |f(x) - 2| < \varepsilon$$

$$(0.4) \quad \left| \frac{x^2 - 1}{x - 1} - 2 \right| < \varepsilon$$

$$(0.5) \quad \left| \frac{(x + 1)(x - 1)}{x - 1} - 2 \right| < \varepsilon$$

$$(0.6) \quad |x + 1 - 2| < \varepsilon$$

$$(0.7) \quad |x - 1| < \varepsilon$$

First, notice that cancelling out the factor of  $(x - 1)$  in the third line is legal here, since we are explicitly assuming that  $x \neq 1$  (even though it can be very, very close to 1) so there is no question of dividing by zero.

Second, notice that if we take  $\delta = \varepsilon$  the last inequality in the above derivation is exactly the condition we want to impose on  $x$ . That is, we have shown that if we restrict  $f(x)$  to be within a distance of  $\varepsilon$ , then  $x$  is forced to be within the same distance of 1. Since the proof is reversible (we can start with the last line and work backwards up to the first line), we can simply invert the steps to get a proof that the limit of  $f(x)$  is 2 as  $x$  approaches 1.

The formal notation for the limit is

$$(0.8) \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

Again, it must be emphasized that saying that the limit of a function  $f(x)$  is some value  $L$  as  $x$  approaches a particular value  $x_0$  is not the same as saying that  $f(x)$  actually *has* a value at  $x = x_0$ . We can't just plug  $x = 1$

into the above formula for  $f(x)$  because this results in a division by zero, so the only way we can give  $f(1)$  a meaningful value is to define  $f(1)$  as the limit of  $f(x)$  as  $x$  approaches 1. This may seem picky, but it is important to recognize when you are dividing by zero and take steps to avoid it.

#### PINGBACKS

Pingback: [Derivatives](#)

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