

POSITION AND MOMENTUM UNIT OPERATORS

It's common in quantum mechanics to convert between the eigenstates of the position and momentum operators, so here's a brief summary of how this is done.

The eigenstates of the position operator are delta functions:

$$|\mathbf{x}\rangle = \delta^{(3)}(\mathbf{x} - \mathbf{x}') \quad (0.1)$$

The eigenstates of the momentum operator are

$$|\mathbf{p}\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{p}\cdot\mathbf{x}} \quad (0.2)$$

In non-relativistic quantum mechanics, the unit operator can be written as an expansion in terms of either position or momentum. That is

$$\mathbf{1} = \int d^3x |\mathbf{x}\rangle \langle \mathbf{x}| \quad (0.3)$$

$$= \int d^3x \delta^{(3)}(\mathbf{x} - \mathbf{x}') \delta^{(3)}(\mathbf{x} - \mathbf{x}'') \quad (0.4)$$

$$= \delta^{(3)}(\mathbf{x}'' - \mathbf{x}') \quad (0.5)$$

For momentum

$$\mathbf{1} = \int d^3p |\mathbf{p}\rangle \langle \mathbf{p}| \quad (0.6)$$

$$= \frac{1}{(2\pi)^3} \int d^3p e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \quad (0.7)$$

$$= \delta^{(3)}(\mathbf{x} - \mathbf{x}') \quad (0.8)$$

Inserting either of these unit operators in a bracket leaves it unchanged, as in

$$\langle \psi(\mathbf{x}) | \phi(\mathbf{x}) \rangle = \int d^3 p \langle \psi(\mathbf{x}) | \mathbf{p} \rangle \langle \mathbf{p} | \phi(\mathbf{x}) \rangle \quad (0.9)$$

$$= \frac{1}{(2\pi)^3} \int d^3 p \left[\int d^3 x \psi^*(\mathbf{x}) e^{i\mathbf{p}\cdot\mathbf{x}} \right] \left[\int d^3 x' \phi(\mathbf{x}') e^{-i\mathbf{p}\cdot\mathbf{x}'} \right] \quad (0.10)$$

$$= \left[\int d^3 x \psi^*(\mathbf{x}) \right] \left[\int d^3 x' \phi(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') \right] \quad (0.11)$$

$$= \int d^3 x \psi^*(\mathbf{x}) \phi(\mathbf{x}) \quad (0.12)$$

$$= \langle \psi(\mathbf{x}) | \phi(\mathbf{x}) \rangle \quad (0.13)$$

We can use these formulas to convert from $|\mathbf{x}\rangle$ to $|\mathbf{p}\rangle$. That is

$$\langle \mathbf{x} | \mathbf{p} \rangle = \int d^3 x' \delta^{(3)}(\mathbf{x} - \mathbf{x}') \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{p}\cdot\mathbf{x}'} \quad (0.14)$$

$$= \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{p}\cdot\mathbf{x}} \quad (0.15)$$

$$\langle \mathbf{p} | \mathbf{x} \rangle = \frac{1}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{x}} \quad (0.16)$$

PINGBACKS

Pingback: Klein-Gordon field in position space