POSITION AND MOMENTUM UNIT OPERATORS

It's common in quantum mechanics to convert between the eigenstates of the position and momentum operators, so here's a brief summary of how this is done.

The eigenstates of the position operator are delta functions:

$$|\mathbf{x}\rangle = \delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right) \tag{0.1}$$

The eigenstates of the momentum operator are

$$|\mathbf{p}\rangle = \frac{1}{(2\pi)^{3/2}}e^{i\mathbf{p}\cdot\mathbf{x}} \tag{0.2}$$

In non-relativistic quantum mechanics, the unit operator can be written as an expansion in terms of either position or momentum. That is

$$\mathbf{1} = \int d^3x\,|\mathbf{x}\rangle\langle\mathbf{x}| \tag{0.3}$$

$$= \int d^3x\,\delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right)\delta^{(3)}\left(\mathbf{x}-\mathbf{x}''\right) \tag{0.4}$$

$$= \delta^{(3)}\left(\mathbf{x}''-\mathbf{x}'\right) \tag{0.5}$$

For momentum

$$\mathbf{1} = \int d^3p\,|\mathbf{p}\rangle\langle\mathbf{p}| \tag{0.6}$$

$$= \frac{1}{(2\pi)^3}\int d^3p\,e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \tag{0.7}$$

$$= \delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right) \tag{0.8}$$

Inserting either of these unit operators in a bracket leaves it unchanged, as in

$$\langle\psi(\mathbf{x})|\phi(\mathbf{x})\rangle=\int d^{3}p\,\langle\psi(\mathbf{x})|\mathbf{p}\rangle\langle\mathbf{p}|\phi(\mathbf{x})\rangle \tag{0.9}$$

$$=\frac{1}{(2\pi)^{3}}\int d^{3}p\left[\int d^{3}x\psi^{*}(\mathbf{x})\,e^{i\mathbf{p}\cdot\mathbf{x}}\right]\left[\int d^{3}x'\phi\left(\mathbf{x}'\right)e^{-i\mathbf{p}\cdot\mathbf{x}'}\right] \tag{0.10}$$

$$=\left[\int d^{3}x\psi^{*}(\mathbf{x})\right]\left[\int d^{3}x'\phi\left(\mathbf{x}'\right)\delta\left(\mathbf{x}-\mathbf{x}'\right)\right] \tag{0.11}$$

$$=\int d^{3}x\psi^{*}(\mathbf{x})\,\phi(\mathbf{x}) \tag{0.12}$$

$$=\langle\psi(\mathbf{x})|\phi(\mathbf{x})\rangle \tag{0.13}$$

We can use these formulas to convert from $|\mathbf{x}\rangle$ to $|\mathbf{p}\rangle$. That is

$$\langle\mathbf{x}|\mathbf{p}\rangle=\int d^{3}x'\delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right)\frac{1}{(2\pi)^{3/2}}e^{i\mathbf{p}\cdot\mathbf{x}'} \tag{0.14}$$

$$=\frac{1}{(2\pi)^{3/2}}e^{i\mathbf{p}\cdot\mathbf{x}} \tag{0.15}$$

$$\langle\mathbf{p}|\mathbf{x}\rangle=\frac{1}{(2\pi)^{3/2}}e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{0.16}$$

PINGBACKS

Pingback: Klein-Gordon field in position space