We’ve seen that we can calculate the eigenfunctions of the momentum operator by solving a simple differential equation. Finding the eigenfunctions of the position operator requires a different approach since no differential equation is involved. Its eigenvalue equation is

$$x |e_{x'}⟩ = x' |e_{x'}⟩$$  \hspace{1cm} (1)

where the $x$ on the LHS is the position operator, even though it just multiplies its eigenfunction, and the $x'$ on the RHS is a constant eigenvalue. Thus we’re looking for an eigenfunction which has the property that when it’s multiplied by $x$ it gives back the same function multiplied by a constant $x'$. At first glance, you might think that we can just set $x = x'$ and this equation is then true for any function. However, remember that we’re trying to find a function such that multiplying it by the continuous variable $x$ results in a fixed value $x'$ multiplied by the same function. If this confuses you, think back to the hamiltonian and its eigenvalue equation $Hψ_n = E_nψ_n$. Here we have a single operator $H$ which, when operating on a particular function $ψ_n$ gives back that same function multiplied by a constant $E_n$. If we give it a different eigenfunction, the same operator will multiply this eigenfunction by a different eigenvalue.

In the case of the position operator, then, we want to find a function that, when operated on by the operator $x$ gives back the same function multiplied by a particular value $x'$. This function can be zero everywhere except at $x = x'$, which leads us to using the delta function as the eigenfunction. This is because

$$x δ(x - x') = x' δ(x - x')$$  \hspace{1cm} (2)

where, remember, the $x$ is a continuous variable and the $x'$ is a constant. The delta function picks out the one value of $x$ where $x = x'$. 

Thus we can say

$$|e_{x'}\rangle = A\delta(x - x')$$  \hspace{1cm} (3)

for some constant $A$.

This function isn’t square integrable, but if we try to calculate the inner product we get

$$\langle e_{x} | e_{y} \rangle = |A|^2 \int \delta(x - z)\delta(x - y)dx$$  \hspace{1cm} (4)

$$= |A|^2 \delta(z - y)$$  \hspace{1cm} (5)

If we pick $A = 1$, we get a sort of pseudo-orthonormality condition:

$$\langle e_{z} | e_{y} \rangle = \delta(z - y)$$  \hspace{1cm} (6)

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