SPIN - INTRODUCTION

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Sec 4.4.

In classical physics, angular momentum is commonly divided into two types: orbital and rotational. In the case of the Earth, for example, the orbital angular momentum is associated with the revolution of the Earth about the Sun, while rotational angular momentum is associated with the Earth rotating on its axis. Rotational angular momentum is often known as spin, for fairly obvious reasons.

In classical physics, of course, there is really only one type of angular momentum, since both the orbital and rotational types are associated with the rotation of masses about some axis. The division into the two types is purely a computational convenience.

In quantum mechanics, the situation is quite different. The orbital angular momentum *is* associated with the motion of masses about an axis (so the quantum angular momentum operator $\mathbf{L}$ is derived from the classical expression $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ by substituting in the quantum operators). The angular momentum of the electron in the hydrogen atom is thus a description of the motion of the electron about the proton, although the planetary model of a solid electron rotating about a solid proton isn't accurate. We can see this since the components of $\mathbf{L}$ don't commute with each other, so they aren't simultaneously observable. Thus it is impossible to specify the vector $\mathbf{L}$ in quantum mechanics; only one of its three components may be given precisely, along with its magnitude (in the form L^2 which does commute with all three of the components of $\mathbf{L}$).

Originally, it was assumed that elementary particles like the electron were solid spheres that could rotate on their axes and thus the notion of a rotational angular momentum or spin could be defined in analogy with the classical case. However, from considerations involving electromagnetism and relativity, a *classical electron radius* can be calculated, which turns out to be

$$r_c = \frac{e^2}{4\pi\varepsilon_0 mc^2} \tag{0.1}$$

where e is the electron charge, m is its mass, c is the speed of light and $1/4\pi\varepsilon_0$ is the Coulomb constant. Plugging in the numbers gives a value of

$r_c = 2.8 \times 10^{-15}$ m. If the spin angular momentum of an electron is $\hbar/2$, we can use the classical formula $L = I\omega$ where I is the moment of inertia and ω is the angular velocity. For a sphere $I = \frac{2}{5}mr^2$ so we get

$$\frac{\hbar}{2} = \frac{2}{5}mr_c^2\omega \tag{0.2}$$

$$\omega = \frac{5\hbar}{4mr_c^2} \tag{0.3}$$

From this, we can get the speed v of a point on the electron's equator from the relation $v = r_c\omega$ and if we plug in the numbers we get

$$v = 3.24 \times 10^{11}\ m/s \tag{0.4}$$

which is of course faster than the speed of light. (In fact, experimentally, the radius of the electron is known to be considerably less than r_c, but this just makes v bigger, so the problem gets worse.) The classical theory of electron spin thus doesn't make any sense when transferred over to the quantum domain. However, the hypothesis that the electron (and other particles) actually *do* have spin was highly successful in explaining many experimental phenomena such as the Zeeman effect (splitting of spectral lines in a magnetic field), so the need for spin in some form is genuine.

The usual approach to quantum spin is just to postulate that elementary particles have an intrinsic spin, and that this spin isn't due to any physical motion of the particle; it just *is*, in much the same way that particles have mass, charge and (in the case of the more esoteric particles) several other quantities such as strangeness. This isn't terribly satisfactory from an intellectual point of view, but since I'm not qualified to write about the details of particle physics, we have to start somewhere.

Thus we propose a theory of spin starting with the basic commutation relations, which we assume are the same as those for the components of $\mathbf{L}$. That is, we begin with a spin vector $\mathbf{S}$ whose components satisfy:

$$[S_x, S_y] = i\hbar S_z \tag{0.5}$$

$$[S_y, S_z] = i\hbar S_x \tag{0.6}$$

$$[S_z, S_x] = i\hbar S_y \tag{0.7}$$

Since the eigenvalues of L^2 and L_z were derived entirely from the commutators, we can arrive at the same conclusions for spin, and thus we find that the eigenvalues of S^2 are $\hbar^2 s(s+1)$ and of S_z (or S_x or S_y) are $\hbar m_s$.

Here s is any non-negative integer or half-integer, and for each value of s, m_s can take on values in integer steps in the range $m_s = -s, -s+1, \ldots, s-1, s$.

The difference between **L** and **S** is that particles can take on various values of L^2, but each elementary particle has a fixed, specific value of spin, so its value of s is a fundamental constant. For example, protons, neutrons and electrons all have spin $s = \frac{1}{2}$, photons have spin $s = 1$ and so on.

Another key difference is that the eigenfunctions of L^2 and L_z are actual functions of spatial position (in fact they are spherical harmonics). The eigenfunctions of S^2 and S_z aren't actually functions at all, really, and it is better to call them eigenstates. At least they don't depend on position, since they represent intrinsic states of elementary particles. It is thus normal to write them in the ket notation, so, for example, the two eigenstates possible for the spin component of an electron's state are

$$|s\, m_s\rangle = \left|\frac{1}{2}\ \frac{1}{2}\right\rangle;\ \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle \tag{0.8}$$

PINGBACKS

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