BINARY OPERATIONS ON SETS

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Post date: 18 September 2025.

Reference: *A Gentle Introduction to Group Theory*, Bana Al Subaiei & Muneerah Al Nuwairan, Section 4.1.

A binary operation, denoted by the generic symbol $*$, is an operation defined on a set G that takes two elements of G and maps them onto another element of G. That is, formally

$$* : G \times G \to G \tag{1}$$

The operation must be a function, which means it must satisfy the following conditions:

(1) For each $a, b \in G$, the operation $*$ assigns an image $a * b \in G$.
(2) The assigned image $a * b$ must be in G.
(3) The assigned image must be unique. That is, if $a, b, c, d \in G$ and $a = c$ and $b = d$, then $a * b = c * d$.

The operation is commutative if $a * b = b * a$ for all $a, b \in G$. It is associative if $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$.

Example 1. Let A be the set of negative integers, and $*$ be subtraction, so that $a * b = a - b$. This is not a binary operation, since if $a = -1$ and $b = -2$, we have $a * b = -1 - (-2) = +1$, which is not in the set A.

Example 2. Let $A = \mathbb{Q}$, the set of rational numbers, and $*$ is subtraction. In this case, a typical operation is of the form

$$\frac{a}{b} * \frac{c}{d} = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} \tag{2}$$

The result on the RHS is also a rational number, so all images of the operation are also in $\mathbb{Q}$.

To test uniqueness, suppose we have $\frac{a}{b} = \frac{1}{2}$, $\frac{c}{d} = \frac{5}{7}$. Then

$$\frac{1}{2} - \frac{5}{7} = \frac{7 - 10}{14} = -\frac{3}{14} \tag{3}$$

Now suppose we have $\frac{a}{b} = \frac{2}{4}$ and $\frac{c}{d} = \frac{10}{14}$, which are numerically the same as the original rational numbers. We then have

$$\frac{2}{4} - \frac{10}{14} = \frac{28-40}{56} = -\frac{12}{56} = -\frac{3}{14} \tag{4}$$

Thus we get the same result as in 3. In general, if we have, with $m, n \in \mathbb{Z}\backslash\{0\}$

$$\frac{na}{nb} * \frac{mc}{md} = \frac{na}{nb} - \frac{mc}{md} \tag{5}$$

$$= \frac{nmad - nmbc}{nmbd} \tag{6}$$

$$= \frac{ad - bc}{bd} \tag{7}$$

Thus the images are unique. Thus this is a binary operation.

Example 3. Let $x, y \in \mathbb{Z}$and $*$ be the operation

$$a * b = xa + yb \tag{8}$$

with $a, b \in \mathbb{R}$ (the real numbers).

We can show that $*$ is associative if and only if $x, y \in \{0, 1\}$. If it's associative, then

$$a * (b * c) = xa + y(xb + yc) \tag{9}$$

$$= xa + xyb + y^2 c \tag{10}$$

$$= (a * b) * c \tag{11}$$

$$= x(xa + yb) + yc \tag{12}$$

$$= x^2 a + xyb + yc \tag{13}$$

Thus we must have

$$xa + xyb + y^2 c = x^2 a + xyb + yc \tag{14}$$

Since a, b, c are arbitrary, we can equate coefficients, giving

$$\begin{aligned} x &= x^2 \\ xy &= xy \\ y^2 &= y \end{aligned} \tag{15}$$

The top and bottom equations require that $x = 0$ or 1 and $y = 0$ or 1.

Now suppose that $x, y \in \{0, 1\}$. From 10 and 13 we have 4 possibilities, shown in Table 1.

In all cases, we have $a * (b * c) = (a * b) * c$.

x	y	$xa+xyb+y^2c$	$x^2a+xyb+yc$
0	0	0	0
0	1	c	c
1	0	a	a
1	1	$a+b+c$	$a+b+c$

TABLE 1. Associative property.

We can also show that $*$ is commutative if and only if $x=y$. If $x=y$, then

$$a*b=xa+xb=x\left(a+b\right) \tag{16}$$

$$b*a=xb+xa=x\left(a+b\right) \tag{17}$$

If $*$ is commutative, then

$$a*b=xa+yb=b*a=xb+ya \tag{18}$$

Again, since a,b are arbitrary, we can equate coefficients, giving $x=y$.

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