CYCLES IN PERMUTATION GROUPS

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Reference: *A Gentle Introduction to Group Theory*, Bana Al Subaiei & Muneerah Al Nuwairan, Section 6.2.

The symmetric group $\mathfrak{S}_n$ on a set of n objects is the set of possible permutations of these objects. A given permutation can be represented in matrix form, but an alternative is to write it as the product of one or more cycles. For example, a permutation in $\mathfrak{S}_5$ in matrix form is

$$\phi=\left(\begin{array}{ccccc}1 & 2 & 3 & 4 & 5\\ 3 & 4 & 1 & 5 & 2\end{array}\right) \tag{1}$$

This can be written as a product of two *cycles* in the form

$$\phi=\left(\begin{array}{cc}1 & 3\end{array}\right)\left(\begin{array}{ccc}2 & 4 & 5\end{array}\right) \tag{2}$$

The interpretation of this is that, in the cycle $\left(\begin{array}{ccc}2 & 4 & 5\end{array}\right)$, each element is permuted to the next element in the list, with the last element moving to the first element. Thus $2\to4\to5\to2$. The cycle $\left(\begin{array}{cc}1 & 3\end{array}\right)$ swaps 1 and 3.

Another example is the permutation given in matrix form as

$$\psi=\left(\begin{array}{ccccc}1 & 2 & 3 & 4 & 5\\ 2 & 3 & 1 & 4 & 5\end{array}\right) \tag{3}$$

As both 4 and 5 are left unchanged, the cycle form of this permutation usually just omits them, so we have

$$\psi=\left(\begin{array}{ccc}1 & 2 & 3\end{array}\right) \tag{4}$$

which is equivalent to

$$\psi=\left(\begin{array}{ccc}1 & 2 & 3\end{array}\right)(4)(5) \tag{5}$$

Converting from cycle form back into matrix form requires tracing the paths through the cycles. For example, consider the permutation β on $\mathfrak{S}_{11}$ given by

$$\beta=\left(\begin{array}{ccc}2 & 5 & 3\end{array}\right)\left(\begin{array}{ccc}8 & 9 & 11\end{array}\right)\left(\begin{array}{ccc}7 & 1 & 4\end{array}\right) \tag{6}$$

To write this in matrix form, we begin with the rightmost cycle and trace the path through the cycles working from right to left. From the last cycle, we have $7\to1\to4\to7$ so we can fill in these entries in the matrix:

$$\beta=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11\\ 4 & & & 7 & & & 1 & & & & \end{pmatrix} \tag{7}$$

Then, in the next cycle we have $8\to9\to11\to8$, so we fill in these entries:

$$\beta=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11\\ 4 & & & 7 & & & 1 & 9 & 11 & & 8 \end{pmatrix} \tag{8}$$

Finally, the leftmost cycle gives us $2\to5\to3\to2$, so we get

$$\beta=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11\\ 4 & 5 & 2 & 7 & 3 & & 1 & 9 & 11 & & 8 \end{pmatrix} \tag{9}$$

Since neither 6 nor 10 is mentioned in 6, these two objects are not moved, so we can fill in the last two spaces in the matrix:

$$\beta=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11\\ 4 & 5 & 2 & 7 & 3 & 6 & 1 & 9 & 11 & 10 & 8 \end{pmatrix} \tag{10}$$

One advantage of cycle notation is that the inverse of a permutation can be written by just reversing the order of the elements. Thus

$$\begin{pmatrix} 2 & 4 & 5 \end{pmatrix}^{-1}=\begin{pmatrix} 5 & 4 & 2 \end{pmatrix} \tag{11}$$

Two cycles are called *disjoint* if they contain no elements in common. Thus $\begin{pmatrix} 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 2 & 4 & 5 \end{pmatrix}$ in 2 are disjoint, but $\begin{pmatrix} 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 5 & 4 \end{pmatrix}$ disjoint cycles are not.

Any permutation of n objects can, in fact, be written as a finite product of disjoint cycles (Proposition 6.2.9 in the above reference). Since disjoint cycles have no elements in common, a product of disjoint cycles commutes. Thus we could write 6 as

$$\beta=\begin{cases} \begin{pmatrix} 2 & 5 & 3 \end{pmatrix}\begin{pmatrix} 8 & 9 & 11 \end{pmatrix}\begin{pmatrix} 7 & 1 & 4 \end{pmatrix} & \text{or}\\ \begin{pmatrix} 2 & 5 & 3 \end{pmatrix}\begin{pmatrix} 7 & 1 & 4 \end{pmatrix}\begin{pmatrix} 8 & 9 & 11 \end{pmatrix} & \text{or}\\ \begin{pmatrix} 7 & 1 & 4 \end{pmatrix}\begin{pmatrix} 8 & 9 & 11 \end{pmatrix}\begin{pmatrix} 2 & 5 & 3 \end{pmatrix} & \text{or}\\ \vdots \end{cases} \tag{12}$$

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