SETS, GROUPS, FIELDS, VECTOR SPACES AND ALGEBRAS

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Although we've met these topics before, it's worth giving an outline of all of them in one place in order to see the connections between them. This summary follows the treatment in chapter 1 of Gilmore's book on Lie groups.

SETS

A *set* is defined as a collection of objects, full stop. The objects in a set need not have any connection to each other. No operations for combining or interacting the objects need be provided. [In most computer languages, a set is not allowed to have more than one copy of each object that it contains, although this restriction isn't mentioned in Gilmore's book.] As sets have no structure beyond the collection of objects, there isn't a lot more we can say about them in a physics context. We won't consider actions such as union, intersection or difference here, although they are important in more general set theory.

A set may contain discrete objects such as the people in a population, or it may contain continuous objects such as the points on the real number line.

GROUPS

We've defined the axioms of a group before, so I'll summarize them here. A group G is a set of objects g_i. An operation, usually called *multiplication* or *composition*, is specified for combining two group elements. The group must satisfy the following axioms:

(1) For any two objects g_i and g_j in a group G, their composition g_ig_j is also in the group. Mathematically, this is written as $\forall g_i,g_j\in G,g_ig_j\in G$. Here the symbol $\forall$ is shorthand for 'for all', and $\in$ means 'is a member of' or just 'in'. This property is known as *closure*.

(2) Any group G must contain an identity element I which obeys the composition rules $g_iI=Ig_i=g_i$. That is, multiplying any group element g_i by the identity I leaves g_i unchanged.

(3) Every element $g_i \in G$ must have an inverse g_i^{-1}, with the property that $g_i^{-1} g_i = g_i g_i^{-1} = I$. That is, the inverse g_i^{-1} 'undoes' whatever the element g_i does.
(4) The composition or multiplication operation must be associative, so that $(g_i g_j) g_k = g_i (g_j g_k)$.

Multiplication need not be commutative, so that in general $g_i g_j \neq g_j g_i$. However, if multiplication *is* commutative for all pairs of group elements, the group is called *abelian*.

A group may be *discrete* if it contains discrete objects, such as the permutations of a set of objects. Conversely, a group may be *continuous* if its elements cover a continuous range. A common example of a continuous group is the group of rotations in two dimensions.

A large number of theorems can be proved starting from the group axioms, many of which we have already covered (see the table of contents).

Fields

Although we've implicitly used fields in much of our studies, I don't think I've actually written down a formal definition of a field. A field F is a set of elements f_i together with two operations: *addition*, symbolized by + as in $f_1 + f_2$, and *scalar multiplication*, symbolized by $\circ$ as in $f_1 \circ f_2$, or sometimes by just writing two field elements together without any operator between them, as in $f_1 f_2$.

The field elements satisfy the following postulates.

Postulate A: F is an abelian group under the + operation. That is, if we take the field addition operation + as the group 'multiplication' operation, then the elements f_i form an abelian group. By convention, the identity element under + is denoted f_0. You can remember this by noting that the identity element must leave all other elements unchanged under the + operation, and adding 0 to anything doesn't change it. The inverse of an element f_i under addition is $-f_i$, since $f_i + (-f_i) = 0 = f_0$.

Postulate B concerns the scalar multiplication operation, and is similar to the group axioms, but with some subtle differences. The postulate consists of the following axioms:

(1) Scalar multiplication of two elements, as in $f_i \circ f_j$ always results in another element of the field F. This is known as *closure*, as in a group.
(2) Associativity: $f_i \circ (f_j \circ f_k) = (f_i \circ f_j) \circ f_k$.
(3) Existence of an identity element, denoted 1, so that $f_i \circ 1 = 1 \circ f_i = f_i$. Note that the identity element under scalar multiplication is *not* the same as the identity f_0 for addition.

(4) Inverses. For every element f_i *except* f_0, there is an inverse element f_i^{-1} such that $f_i f_i^{-1} = f_i^{-1} f_i = 1$. The exclusion of f_0 is due to the fact that 0 has no inverse.
(5) Distributive law: $f_i \circ (f_j + f_k) = f_i \circ f_j + f_i \circ f_k$, and $(f_i + f_j) \circ f_k = f_i \circ f_k + f_j \circ f_k$. Note that both operations + and $\circ$ are used in this law.

Note that scalar multiplication is *not* assumed to be commutative, but if it is, the field is called a *commutative field*. Note that a field is *always* commutative under addition, but not necessarily under scalar multiplication.

The two most common fields in physics are the real and complex numbers, both of which are commutative. The quaternions also constitute a field, but since quaternion multiplication is not commutative, the quaternion field is not a commutative field.

LINEAR VECTOR SPACES

We've also looked in detail at vector spaces, so again I'll just summarize their definition here. A linear (henceforth, we'll be concerned only with linear vector spaces, so I'll drop the term 'linear' to save space) vector space consists of a set V of objects $\boldsymbol{v}_i$ called *vectors*, and a field F. Two kinds of operations are defined for a vector space: vector addition, symbolized by + as in $\boldsymbol{v}_i + \boldsymbol{v}_j$, and scalar multiplication, symbolized by $\circ$ as in $f_i \circ \boldsymbol{v}_j$ or sometimes without an operator, as in $f_i \boldsymbol{v}_j$.

The postulates or axioms of a vector space are similar to those for a field, but since we're dealing with two different types of objects, they're not quite the same thing.

Postulate A is that V is an abelian group under addition. The identity vector is denoted $\boldsymbol{v}_0$ or sometimes as $\mathbf{0}$ (a bold zero). As with the field identity f_0, $\boldsymbol{v}_0$ can be thought of as adding nothing, thus leaving any other vector unchanged. The inverse of $\boldsymbol{v}_i$ is $-\boldsymbol{v}_i$, since $\boldsymbol{v}_i + (-\boldsymbol{v}_i) = \mathbf{0} = \boldsymbol{v}_0$.

Postulate B concerns the action of the field elements on the vectors. Its axioms are

(1) For all $f_i \in F$ and $\boldsymbol{v}_j \in V$, $f_i \circ \boldsymbol{v}_j = f_i \boldsymbol{v}_j \in V$. That is, multiplying any vector by any scalar results in another vector also in the vector space. This is *closure*.
(2) Associativity: $f_i \circ (f_j \circ \boldsymbol{v}_k) = (f_i \circ f_j) \circ \boldsymbol{v}_k$.
(3) Identity element 1 in the field F. $1 \circ \boldsymbol{v}_i = \boldsymbol{v}_i \circ 1 = \boldsymbol{v}_i$. This identity element is the same as the identity element in the field F under scalar multiplication, but *not* the identity for addition.
(4) Bilinearity: $f_i \circ (\boldsymbol{v}_k + \boldsymbol{v}_\ell) = f_i \circ \boldsymbol{v}_k + f_i \circ \boldsymbol{v}_\ell$, and $(f_i + f_j) \circ \boldsymbol{v}_k = f_i \circ \boldsymbol{v}_k + f_j \circ \boldsymbol{v}_k$.

Some examples of vector spaces have been given earlier. In particular, it's worth reemphasizing here that whether a vector space is 'real' or 'complex' is determined by the choice of field F, and *not* by whether the vectors $\boldsymbol{v}_i$ contain real or complex numbers.

Further articles on vector spaces and linear algebra, especially as applied to quantum mechanics, are in my table of contents.

ALGEBRAS

This use of the term 'algebra' is not that which is usually encountered in high school mathematics classes, where algebra refers to the solution of equations with variables specified by symbols such as x and y. The algebra we're concerned with here is more properly called *linear algebra* and is essentially a linear vector space with an extra form of multiplication defined. That is:

A linear algebra contains a set V of vectors $\{\boldsymbol{v}_i\}$ and a field F. The three operations in an algebra are vector addition (denoted +) and scalar multiplication (denoted $\circ$), both of which are the same as for a vector space, and in addition a vector multiplication operation (denoted $\square$). All the postulates of a vector space concerning addition + and scalar multiplication $\circ$ carry over to an algebra, but there are some additional axioms concerning the vector multiplication $\square$, as follows.

(1) Closure: $\boldsymbol{v}_i\square\boldsymbol{v}_j \in V$.
(2) Bilinearity: $(\boldsymbol{v}_i+\boldsymbol{v}_j)\square\boldsymbol{v}_k = \boldsymbol{v}_i\square\boldsymbol{v}_k+\boldsymbol{v}_j\square\boldsymbol{v}_k$, and $\boldsymbol{v}_i\square(\boldsymbol{v}_j+\boldsymbol{v}_k) = \boldsymbol{v}_i\square\boldsymbol{v}_j+\boldsymbol{v}_i\square\boldsymbol{v}_k$.

There are other axioms that may or may not apply to any given algebra. These are

(3) Associativity: $(\boldsymbol{v}_i\square\boldsymbol{v}_j)\square\boldsymbol{v}_k = \boldsymbol{v}_i\square(\boldsymbol{v}_j\square\boldsymbol{v}_k)$.
(4) Identity: $\boldsymbol{v}_i\square\mathbf{1} = \boldsymbol{v}_i$. The **1** vector need not be the same identity as that for + or $\circ$.
(5) Symmetry/antisymmetry: $\boldsymbol{v}_i\square\boldsymbol{v}_j = \pm\boldsymbol{v}_j\square\boldsymbol{v}_i$.
(6) Derivative: $\boldsymbol{v}_i\square(\boldsymbol{v}_j\square\boldsymbol{v}_k) = (\boldsymbol{v}_i\square\boldsymbol{v}_j)\square\boldsymbol{v}_k+\boldsymbol{v}_j\square(\boldsymbol{v}_i\square\boldsymbol{v}_k)$.

As an example, the set of $n\times n$ matrices constitutes an algebra if + is matrix addition, $\circ$ is scalar multiplication and $\square$ is matrix multiplication.

The + operation adds corresponding entries of two matrices A and B, so is essentially scalar addition performed n^2 times and thus matrix addition satisfies the conditions such as existence of an inverse ($B=-A$), identity (matrix with all entries 0), and so on.

Scalar multiplication multiplies each element of a matrix by the same scalar, so is essentially scalar multiplication performed n^2 times.

To show that the set is an algebra, we need to show that it satisfies axioms 1 and 2 of an algebra as stated above. Closure is verified by noting that the product of 2 $n \times n$ matrices is always another $n \times n$ matrix.

To show bilinearity, we can write out the expression using components and the summation convention, so that k is summed.

$$[(A+B)\square C]_{ij} = (A_{ik}+B_{ik})\,C_{kj} \tag{1}$$
$$= A_{ik}C_{kj}+B_{ik}C_{kj} \tag{2}$$
$$= [A\square C+B\square C]_{ij} \tag{3}$$

The same process verifies the second condition in axiom 2.

Matrix multiplication has an identity element in the identity matrix, so axiom 4 is satisfied. However, in general, matrix multiplication is not commutative, so axiom 5 is *not* satisfied.