HAMILTON'S EQUATIONS AND POISSON BRACKETS

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References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 2.

Because Klauber's approach to QFT depends on generalizing classical mechanics, it's worth seeing how the basic equations of classical mechanics are derived. We've already seen that the Euler-Lagrange equation is derived from the principle of least action using the calculus of variations. The equation for a single particle in one dimension is

$$\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}}=0 \tag{1}$$

where L is the Lagrangian

$$L=L(q,\dot{q})=T-V \tag{2}$$

We can generalize this to a system with d degrees of freedom (the number of degrees of freedom is the number of particles multiplied by the number of dimensions) as

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}=0 \tag{3}$$

where $i=1,\ldots,d$. Solving this system of differential equations gives the particle trajectories as functions of time.

The Euler-Lagrange equations can be put into a different form by means of a *Legendre transformation* as follows. We define the *conjugate momenta* as

$$p_k\equiv\frac{\partial L}{\partial\dot{q}_k} \tag{4}$$

We also define the *Hamiltonian*

$$H\equiv\sum_k p_k\dot{q}_k-L \tag{5}$$

Taking derivatives of H gives (treating p_k and q_k as the independent variables):

$$\frac{\partial H}{\partial p_i} = \dot{q}_i + \sum_k p_k \frac{\partial \dot{q}_k}{\partial p_i} - \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial p_i} \tag{6}$$

$$= \dot{q}_i \tag{7}$$

where we used 4 to cancel off the last two sums.

Similarly, we get

$$\frac{\partial H}{\partial q_i} = \sum_k p_k \frac{\partial \dot{q}_k}{\partial q_k} - \frac{\partial L}{\partial q_i} - \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial q_i} \tag{8}$$

$$= -\frac{\partial L}{\partial q_i} \tag{9}$$

Comparing this with 4 and 3 we see that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = -\frac{\partial H}{\partial q_i} - \frac{d}{dt} p_i \tag{10}$$

$$= 0 \tag{11}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{12}$$

We thus get Hamilton's equations which are equivalent to the Euler-Lagrange equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{13}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{14}$$

For a general function $u(q_i, p_i, t)$ of the generalized coordinates q_i, conjugate momenta p_i and time t, its time derivative is

$$\frac{du}{dt} = \sum_k \frac{\partial u}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial u}{\partial p_k}\dot{p}_k + \frac{\partial u}{\partial t} \tag{15}$$

$$= \sum_k \left(\frac{\partial u}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial H}{\partial q_k}\right) + \frac{\partial u}{\partial t} \tag{16}$$

The sum in the last line is called the *Poisson bracket* and written as

$$\{u,H\} \equiv \sum_k \left(\frac{\partial u}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial H}{\partial q_k} \right) \tag{17}$$

Hamilton's equations can be written in terms of Poisson brackets (remember that the independent variables are p_i and q_i):

$$\begin{aligned}
\dot{p}_i &= \{p_i,H\} && (18)\\
&= \sum_k \left(\frac{\partial p_i}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial H}{\partial q_k} \right) && (19)\\
&= -\frac{\partial H}{\partial q_i} && (20)\\
\dot{q}_i &= \{q_i,H\} && (21)\\
&= \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial H}{\partial q_k} \right) && (22)\\
&= \frac{\partial H}{\partial p_i} && (23)
\end{aligned}$$

Finally, we'll have a look at the Poisson brackets for conjugate variables:

$$\begin{aligned}
\{q_i,p_j\} &= \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} \right) && (24)\\
&= \delta_{ij} && (25)\\
\{q_i,q_j\} &= \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k} \right) && (26)\\
&= 0 && (27)\\
\{p_i,p_j\} &= \sum_k \left(\frac{\partial p_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} \right) && (28)\\
&= 0 && (29)
\end{aligned}$$

The Poisson brackets for position q_i and momentum p_i bear an uncanny resemblance to the commutators of position and momentum in quantum mechanics:

$$\begin{aligned}
[x_i,p_j] &= i\delta_{ij} && (30)\\
[x_i,x_j] &= 0 && (31)\\
[p_i,p_j] &= 0 && (32)
\end{aligned}$$

In fact, the correspondence between Poisson brackets and commutators was used in the development of quantum mechanics as a guide to quantizing classical theory.

PINGBACKS

Pingback: Hamilton's equations for relativistic fields; conjugate momentum

Pingback: Poisson brackets in classical field theory

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