

HAMILTON'S EQUATIONS AND POISSON BRACKETS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 2.

Because Klauber's approach to QFT depends on generalizing classical mechanics, it's worth seeing how the basic equations of classical mechanics are derived. We've already seen that the Euler-Lagrange equation is derived from the principle of least action using the calculus of variations. The equation for a single particle in one dimension is

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0 \quad (1)$$

where L is the Lagrangian

$$L = L(q, \dot{q}) = T - V \quad (2)$$

We can generalize this to a system with d degrees of freedom (the number of degrees of freedom is the number of particles multiplied by the number of dimensions) as

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \quad (3)$$

where $i = 1, \dots, d$. Solving this system of differential equations gives the particle trajectories as functions of time.

The Euler-Lagrange equations can be put into a different form by means of a *Legendre transformation* as follows. We define the *conjugate momenta* as

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k} \quad (4)$$

We also define the *Hamiltonian*

$$H \equiv \sum_k p_k \dot{q}_k - L \quad (5)$$

Taking derivatives of H gives (treating p_k and q_k as the independent variables):

$$\frac{\partial H}{\partial p_i} = \dot{q}_i + \sum_k p_k \frac{\partial \dot{q}_k}{\partial p_i} - \sum_k \frac{\partial L}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial p_i} \quad (6)$$

$$= \dot{q}_i \quad (7)$$

where we used 4 to cancel off the last two sums.

Similarly, we get

$$\frac{\partial H}{\partial q_i} = \sum_k p_k \frac{\partial \dot{q}_k}{\partial q_i} - \frac{\partial L}{\partial q_i} - \sum_k \frac{\partial L}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial q_i} \quad (8)$$

$$= -\frac{\partial L}{\partial q_i} \quad (9)$$

Comparing this with 4 and 3 we see that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = -\frac{\partial H}{\partial q_i} - \frac{d}{dt} p_i \quad (10)$$

$$= 0 \quad (11)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (12)$$

We thus get Hamilton's equations which are equivalent to the Euler-Lagrange equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (13)$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad (14)$$

For a general function $u(q_i, p_i, t)$ of the generalized coordinates q_i , conjugate momenta p_i and time t , its time derivative is

$$\frac{du}{dt} = \sum_k \frac{\partial u}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial u}{\partial p_k} \dot{p}_k + \frac{\partial u}{\partial t} \quad (15)$$

$$= \sum_k \left(\frac{\partial u}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial H}{\partial q_k} \right) + \frac{\partial u}{\partial t} \quad (16)$$

The sum in the last line is called the *Poisson bracket* and written as

$$\{u, H\} \equiv \sum_k \left(\frac{\partial u}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial H}{\partial q_k} \right) \quad (17)$$

Hamilton's equations can be written in terms of Poisson brackets (remember that the independent variables are p_i and q_i):

$$\dot{p}_i = \{p_i, H\} \quad (18)$$

$$= \sum_k \left(\frac{\partial p_i}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial H}{\partial q_k} \right) \quad (19)$$

$$= -\frac{\partial H}{\partial q_i} \quad (20)$$

$$\dot{q}_i = \{q_i, H\} \quad (21)$$

$$= \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial H}{\partial q_k} \right) \quad (22)$$

$$= \frac{\partial H}{\partial p_i} \quad (23)$$

Finally, we'll have a look at the Poisson brackets for conjugate variables:

$$\{q_i, p_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) \quad (24)$$

$$= \delta_{ij} \quad (25)$$

$$\{q_i, q_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k} \right) \quad (26)$$

$$= 0 \quad (27)$$

$$\{p_i, p_j\} = \sum_k \left(\frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) \quad (28)$$

$$= 0 \quad (29)$$

The Poisson brackets for position q_i and momentum p_i bear an uncanny resemblance to the commutators of position and momentum in quantum mechanics:

$$[x_i, p_j] = i\delta_{ij} \quad (30)$$

$$[x_i, x_j] = 0 \quad (31)$$

$$[p_i, p_j] = 0 \quad (32)$$

In fact, the correspondence between Poisson brackets and commutators was used in the development of quantum mechanics as a guide to quantizing classical theory.

PINGBACKS

Pingback: Hamilton's equations for relativistic fields; conjugate momentum

Pingback: Poisson brackets in classical field theory

Pingback: Hamilton's equations of motion in classical field theory

Pingback: One-dimensional field (displacement of a string)