HAMILTON'S EQUATIONS FOR RELATIVISTIC FIELDS; CONJUGATE MOMENTUM

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References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 2.

In classical particle theory, we can get the equations of motion of a particle by solving the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=0 \tag{1}$$

where L is the Lagrangian and q_i are the generalized coordinates which describe the location of the particle and thus are, in general, functions of the independent variable t (the time). An alternative form of these equations is given by Hamilton's equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{2}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{3}$$

where the Hamiltonian is defined as

$$H \equiv \sum_k p_k \dot{q}_k - L \tag{4}$$

and the conjugate momentum is

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k} \tag{5}$$

In particle theory, the conjugate momentum is the same as the actual physical momentum of the particle. For a single particle moving in a potential $V(q_k)$ which doesn't depend on the velocity components $\dot{q}_k$, for example, we have

$$L = T-V=\sum_k \frac{1}{2}m\dot{q}_k^2-V(q_k) \tag{6}$$

$$p_k = \frac{\partial L}{\partial \dot{q}_k}=m\dot{q}_k \tag{7}$$

so that p_k is just the component of the momentum **p** in the q_k direction. [In the standard 3-d rectangular system, $q_1 = x$, $q_2 = y$ and $q_3 = z$.]

In classical field theory, spatial coordinates and time are given an equal footing, in the sense that together they serve as labels for particular points in spacetime. There are no particles moving through space, there are only field values at each point in spacetime. Thus the field ϕ depends on q_k but the q_k now include both space *and* time coordinates, and all these coordinates are independent of each other. Thus $\partial q_k/\partial q_j = \delta_{kj}$. The only things that changes in a field theory are the values of the field as we move through spacetime.

One consequence of this is that partial and total derivatives of the field with respect to any of the spacetime coordinates q^k are the same. That is, if we try to use the chain rule to calculate a total time derivative, we get

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial q^k}\frac{\partial q^k}{\partial t} + \frac{\partial\phi}{\partial t} \tag{8}$$

However, $\frac{\partial q^k}{\partial t} = 0$ in field theory because the spatial coordinates q^k do not depend on time; they are merely labels of fixed points in space which don't move.

The Euler-Lagrange equations for classical field theory are

$$\frac{\partial\mathcal{L}}{\partial\phi^r} - \frac{\partial}{\partial q^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^r_{,\mu}}\right) = 0 \tag{9}$$

where $\mathcal{L}$ is now the Lagrangian *density* (the Lagrangian per unit volume), and the superscript r labels which field we're talking about. The index μ runs from 0 to 3 with $q^0 = t$ as usual in relativity.

We can write down a Hamiltonian form of these equations by analogy with the particle theory above by defining the conjugate momentum density

$$\pi_r \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}^r} \tag{10}$$

The Hamiltonian density is then

$$\mathcal{H} = \pi_r\dot{\phi}^r - \mathcal{L} \tag{11}$$

In Klauber's Wholeness Chart 2-2, he states Hamilton's equations of motion for relativistic fields without (as far as I can see) any derivation. The derivation isn't exactly trivial, and doesn't seem to be widely available, so here's how it's done. We can derive Hamilton's equations of motion for the field theory by using the minimum action principle we used to derive the Euler-Lagrange equations in the first place. We get

$$\begin{aligned} S &= \int_{\Omega} \mathcal{L} d^4 q & (12) \\ &= \int_{\Omega} \left(\pi_r \dot{\phi}^r - \mathcal{H}\right) d^4 q & (13) \\ &= \int_{\Omega} \left(\pi_r \partial_0 \phi^r - \mathcal{H}\right) d^4 q & (14) \end{aligned}$$

Varying the action we get

$$\delta S = \int_{\Omega} \delta \left(\pi_r \partial_0 \phi^r - \mathcal{H}\right) d^4 q \tag{15}$$

Looking at the integrand on its own, we get (remember it's only the fields ϕ^r, their derivatives $\phi^r_{,\mu}$ and the conjugate momenta π_r that are variable in the variation)

$$\begin{aligned} \delta \left(\pi_r \partial_0 \phi^r - \mathcal{H}\right) &= \left(\delta \pi_r\right)\left(\partial_0 \phi^r\right) + \pi_r \partial_0 \left(\delta \phi^r\right) - \\ & \quad \frac{\partial \mathcal{H}}{\partial \pi_r} \delta \pi_r - \frac{\partial \mathcal{H}}{\partial \phi^r} \delta \phi^r - \frac{\partial \mathcal{H}}{\partial \phi^r_{,\mu}} \delta \phi^r_{,\mu} & (16) \end{aligned}$$

Since π_r and ϕ^r are treated as independent variables, the variation of each of these variables must be zero on its own. For $\delta \pi_r$ we combine the first and third terms to get

$$\begin{aligned} \left[\partial_0 \phi^r - \frac{\partial \mathcal{H}}{\partial \pi_r}\right] \delta \pi_r &= 0 & (17) \\ \partial_0 \phi^r = \dot{\phi}^r &= \frac{\partial \mathcal{H}}{\partial \pi_r} & (18) \end{aligned}$$

This is the first Hamilton equation of motion, and is analogous to the similar equation from the particle theory: $\dot{q}_i = \frac{\partial H}{\partial p_i}$. The second equation isn't quite as straightforward, however. Returning to 16, we must now deal with the second, fourth and fifth terms. The second term can be written using the product rule as:

$$\pi_r \partial_0 \left(\delta \phi^r\right) = \partial_0 \left[\pi_r \left(\delta \phi^r\right)\right] - \left(\partial_0 \pi_r\right) \delta \phi^r \tag{19}$$

The first term on the RHS is a total derivative with respect to time, and when we integrate this over the time coordinate in 15 we get the term $\pi_r \left(\delta \phi^r\right)$ evaluated at times on the boundary of the four-volume Ω. However, the fields and conjugate momenta on the boundary are fixed, so $\delta \phi^r = 0$ on the boundary, and the integral of this term is zero. We can therefore replace the second term in 16 as follows:

$$\pi_r \partial_0 (\delta\phi^r) \mapsto -(\partial_0 \pi_r)\delta\phi^r \tag{20}$$

We can transform the fifth term in 16 as follows:

$$-\frac{\partial\mathcal{H}}{\partial\phi^r_{,\mu}}\delta\phi^r_{,\mu} = -\partial_\mu\left[\frac{\partial\mathcal{H}}{\partial\phi^r_{,\mu}}\delta\phi^r\right] + \partial_\mu\left(\frac{\partial\mathcal{H}}{\partial\phi^r_{,\mu}}\right)\delta\phi^r \tag{21}$$

The first term on the RHS here is a four-divergence which is integrated over the volume Ω. This can be converted to a surface integral using the 4-d Gauss's theorem, so again gives zero because $\delta\phi^r = 0$ on the boundary. We can therefore replace the fifth term in 16 as follows.

$$-\frac{\partial\mathcal{H}}{\partial\phi^r_{,\mu}}\delta\phi^r_{,\mu} \mapsto \partial_\mu\left(\frac{\partial\mathcal{H}}{\partial\phi^r_{,\mu}}\right)\delta\phi^r \tag{22}$$

Expanding this, we get

$$\partial_\mu\left(\frac{\partial\mathcal{H}}{\partial\phi^r_{,\mu}}\right) = \partial_0\left(\frac{\partial\mathcal{H}}{\partial\phi^r_{,0}}\right) + \partial_i\left(\frac{\partial\mathcal{H}}{\partial\phi^r_{,i}}\right) \tag{23}$$

The first term on the RHS is, using 10 and 11

$$\begin{aligned}
\partial_0\left(\frac{\partial\mathcal{H}}{\partial\phi^r_{,0}}\right) &= \partial_0\left(\frac{\partial\mathcal{H}}{\partial\dot{\phi}^r}\right) && (24)\\
&= \dot{\pi}_r - \partial_0\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}^r}\right) && (25)\\
&= \dot{\pi}_r - \partial_0\pi_r && (26)\\
&= \dot{\pi}_r - \dot{\pi}_r = 0 && (27)
\end{aligned}$$

Combining this with 20 and 22 and inserting into 16, we get for the $\delta\phi^r$ terms:

$$\pi_r\partial_0(\delta\phi^r) - \frac{\partial\mathcal{H}}{\partial\phi^r}\delta\phi^r - \frac{\partial\mathcal{H}}{\partial\phi^r_{,\mu}}\delta\phi^r_{,\mu} \mapsto \left[-\partial_0\pi_r - \frac{\partial\mathcal{H}}{\partial\phi^r} + \partial_i\left(\frac{\partial\mathcal{H}}{\partial\phi^r_{,i}}\right)\right]\delta\phi^r \tag{28}$$

Requiring this variation to be zero gives the second Hamilton's equation:

$$\partial_0\pi_r = \dot{\pi}_r = -\frac{\partial\mathcal{H}}{\partial\phi^r} + \partial_i\left(\frac{\partial\mathcal{H}}{\partial\phi^r_{,i}}\right) \tag{29}$$

Note that the index i in the last term is summed over only the spatial coordinates $i = 1,2,3$. Because of this additional term, this equation of motion is different from its particle analogue $\dot{p}_i = -\frac{\partial H}{\partial q_i}$.

What exactly *is* the conjugate momentum density for a field, in a physical sense? To get an idea, we need to distinguish between the conjugate momentum density of a field as defined by 10 and the *physical* momentum density. In a particle theory, the conjugate momentum of a particle is the same thing as the physical momentum, so that the term $p_i\dot{q}_i$ in the Hamiltonian is just the physical momentum multiplied by the physical velocity, and thus has the dimensions of energy, as required.

The physical velocity of a particle is easy enough to understand, but what is the physical velocity of a field? Suppose we have some scalar field (scalar, to make things simpler) such as the temperature in some volume. The velocity of the field at some point specified by coordinates q^i is the velocity at which we need to move in order to see a constant field value, that is, a constant temperature. For example, if we're in a room and there is a heater against one wall of the room, the temperature at a given point gradually increases as the heater heats the air. However, the location at which the temperature is some fixed value moves steadily away from the heater, and the rate at which this location moves is the velocity of the field at that point. We can call the coordinates of this location x^i which now *are* functions of time (unlike the q^i which always label the same fixed position in spacetime), so the velocity of the field is $v^i = \frac{dx^i}{dt}$. The physical momentum density p^i is then defined so that the Hamiltonian can be written as

$$\mathcal{H} = \sum_i p_i v^i - \mathcal{L} \tag{30}$$

If we define

$$p_i = \pi_r \frac{\partial \phi^r}{\partial x^i} = -\pi_r \frac{\partial \phi^r}{\partial x_i} \tag{31}$$

$$p^i = -\pi_r \frac{\partial \phi^r}{\partial x^i} = \pi_r \frac{\partial \phi^r}{\partial x_i} \tag{32}$$

then 30 reduces to 11 using the chain rule.

It's difficult to interpret this relation physically, because we don't really have a feel for what the conjugate momentum π_r represents physically.

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