POISSON BRACKETS IN CLASSICAL FIELD THEORY

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References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 2.

We've seen that the equations of motion of a particle in classical particle theory can be written in terms of Poisson brackets. For a general function $u(q_i, p_i, t)$ of the generalized coordinates q_i, conjugate momenta p_i and time t, its time derivative is

$$\begin{aligned}\frac{du}{dt} &= \sum_k \left(\frac{\partial u}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial H}{\partial q_k}\right) + \frac{\partial u}{\partial t} &(1)\\ &= \{u, H\} + \frac{\partial u}{\partial t} &(2)\end{aligned}$$

where the Poisson bracket is defined as

$$\{u, H\} \equiv \sum_k \left(\frac{\partial u}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial H}{\partial q_k}\right) \tag{3}$$

To extend this sort of analysis to classical field theory, we replace the generalized coordinates and momenta by the fields ϕ^r and their conjugate momenta π_r. Then, a general function u becomes a function of these new variables $u(\phi^r, \pi_r t)$ so its time derivative is now (implied sum over r):

$$\frac{du}{dt} = \frac{\partial u}{\partial \phi^r}\dot{\phi}^r + \frac{\partial u}{\partial \pi_r}\dot{\pi}_r + \frac{\partial u}{\partial t} \tag{4}$$

Actually, since we're taking the derivative of u with respect to *functions* ϕ^r and π_r, these are really functional derivatives, so we should write this as

$$\frac{du}{dt} = \frac{\delta u}{\delta \phi^r}\dot{\phi}^r + \frac{\delta u}{\delta \pi_r}\dot{\pi}_r + \frac{\partial u}{\partial t} \tag{5}$$

From Hamilton's equations for fields

$$\begin{aligned}\dot{\phi}^r &= \frac{\partial \mathcal{H}}{\partial \pi_r} \equiv \frac{\delta \mathcal{H}}{\delta \pi_r} &(6)\\ \dot{\pi}_r &= -\frac{\partial \mathcal{H}}{\partial \phi^r} + \partial_i\left(\frac{\partial \mathcal{H}}{\partial \phi^r_{,i}}\right) \equiv -\frac{\delta \mathcal{H}}{\delta \phi^r} &(7)\end{aligned}$$

we have the equation of motion for u:

$$\frac{du}{dt} = \frac{\delta u}{\delta\phi^r}\frac{\delta\mathcal{H}}{\delta\pi_r} - \frac{\delta u}{\delta\pi_r}\frac{\delta\mathcal{H}}{\delta\phi^r} + \frac{\partial u}{\partial t} \tag{8}$$

The derivatives $\frac{\partial u}{\partial\phi^r}$ and $\frac{\partial u}{\partial\pi_r}$ require a bit of care. The field ϕ^r and its conjugate momentum π_r are each defined at every point in space, and when we take the derivative of a function u with respect to a field ϕ^r, we get a (possibly) non-zero result only if we take the position **x** of the field in u to be the same as the position of the field with respect to which we're taking the derivative. This is easier to see if we consider 3-space to be divided up into a bunch of infinitesimal volume elements ΔV_j where j is an index specifying which volume element we're in. The field ϕ^r has a separate value in each volume element (and if we consider the elements to be small enough, the field value can be considered constant within each element). We name the field ϕ^r_j to be the field in element ΔV_j. Then we can write u_i as the function u within volume element ΔV_i. Then

$$\frac{\delta u_i}{\delta\phi^r_j} = \delta_{ij}\frac{\delta u_i}{\delta\phi^r_i} \tag{9}$$

Given this, we can write the derivative as a sum over volume elements:

$$\frac{\delta u_i}{\delta\phi^r_i} = \sum_j \frac{\delta u_i}{\delta\phi^r_j} \tag{10}$$

$$= \sum_j \Delta V_j \left[\frac{1}{\Delta V_j}\frac{\delta u_i}{\delta\phi^r_j}\right] \tag{11}$$

In the limit of infinitesimal volume elements, the LHS becomes the derivative evaluated at a particular point **x**, and the volume element in the sum on the RHS becomes the integration element $d^3\mathbf{x}'$. Therefore, the quantity in square brackets must tend to

$$\frac{1}{\Delta V_j}\frac{\delta u_i}{\delta\phi^r_j} \to \delta^3\left(\mathbf{x}-\mathbf{x}'\right)\left.\frac{\delta u\left(\phi^r,\pi_r,t\right)}{\delta\phi^r}\right|_{\phi^r=\phi(\mathbf{x},t)} \equiv \frac{\delta u\left(\phi^r\left(\mathbf{x},t\right),\pi_r\left(\mathbf{x}',t\right)\right)}{\delta\phi^r\left(\mathbf{x}',t\right)} \tag{12}$$

We can do a similar calculation for $\frac{\delta u}{\delta\pi_r}$ with the result

$$\frac{\delta u}{\delta\pi_r} = \delta^3\left(\mathbf{x}-\mathbf{x}'\right)\left.\frac{\delta u\left(\phi^r,\pi_r,t\right)}{\delta\pi^r}\right|_{\pi^r=\pi(\mathbf{x},t)} \equiv \frac{\delta u\left(\phi^r\left(\mathbf{x}',t\right),\pi_r\left(\mathbf{x},t\right)\right)}{\delta\pi^r\left(\mathbf{x}',t\right)} \tag{13}$$

So we get

$$\frac{du(\mathbf{x})}{dt} = \int d^3\mathbf{x}''\left(\frac{\delta u}{\delta\phi^r}\frac{\delta\mathcal{H}}{\delta\pi_r} - \frac{\delta u}{\delta\pi_r}\frac{\delta\mathcal{H}}{\delta\phi^r}\right) + \frac{\partial u}{\partial t} \tag{14}$$

$$= \int d^3\mathbf{x}''\delta^3\left(\mathbf{x}-\mathbf{x}''\right)\delta^3\left(\mathbf{x}'-\mathbf{x}''\right)\left(\frac{\delta u}{\delta\phi^r}\frac{\delta\mathcal{H}}{\delta\pi_r} - \frac{\delta u}{\delta\pi_r}\frac{\delta\mathcal{H}}{\delta\phi^r}\right) + \frac{\partial u}{\partial t} \tag{15}$$

$$= \left(\frac{\delta u}{\delta\phi^r}\frac{\delta\mathcal{H}}{\delta\pi_r} - \frac{\delta u}{\delta\pi_r}\frac{\delta\mathcal{H}}{\delta\phi^r}\right)\delta^3\left(\mathbf{x}-\mathbf{x}'\right) + \frac{\partial u}{\partial t} \tag{16}$$

The Poisson bracket for fields is the first term:

$$\{u,\mathcal{H}\} = \left(\frac{\delta u}{\delta\phi^r}\frac{\delta\mathcal{H}}{\delta\pi_r} - \frac{\delta u}{\delta\pi_r}\frac{\delta\mathcal{H}}{\delta\phi^r}\right)\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{17}$$

For a field and its conjugate momentum, we have

$$\left\{\phi^r\left(\mathbf{x}',t\right),\pi_r\left(\mathbf{x},t\right)\right\} = \int d^3\mathbf{x}''\delta^3\left(\mathbf{x}-\mathbf{x}''\right)\delta^3\left(\mathbf{x}'-\mathbf{x}''\right)\left(\frac{\delta\phi^r}{\delta\phi^r}\frac{\delta\pi_r}{\delta\pi_r} - \frac{\delta\phi^r}{\delta\pi_r}\frac{\delta\pi_r}{\delta\phi^r}\right) \tag{18}$$

$$= \int d^3\mathbf{x}''\delta^3\left(\mathbf{x}-\mathbf{x}''\right)\delta^3\left(\mathbf{x}'-\mathbf{x}''\right)(1-0) \tag{19}$$

$$= \delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{20}$$

If we consider two different fields ϕ^r and ϕ^s, these are independent variables so $\frac{\delta\phi^r}{\delta\phi^s} = \frac{\delta\pi_s}{\delta\pi_r} = \delta^r_s$ and we can generalize the bracket to

$$\left\{\phi^r\left(\mathbf{x}',t\right),\pi_r\left(\mathbf{x},t\right)\right\} = \delta^r_s\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{21}$$

Other brackets are zero, for example

$$\left\{\phi^r\left(\mathbf{x}',t\right),\phi^r\left(\mathbf{x},t\right)\right\} = \int d^3\mathbf{x}''\delta^3\left(\mathbf{x}-\mathbf{x}''\right)\delta^3\left(\mathbf{x}'-\mathbf{x}''\right)\left(\frac{\delta\phi^r}{\delta\phi^r}\frac{\delta\phi^r}{\delta\pi_r} - \frac{\delta\phi^r}{\delta\pi_r}\frac{\delta\phi^r}{\delta\phi^r}\right) \tag{22}$$

$$= 0 \tag{23}$$

because $\frac{\partial\phi^r}{\partial\pi_r} = 0$.

Hamilton's equations of motion for fields can be written using Poisson brackets. We get

$$\frac{d\phi^r}{dt} = \dot{\phi}^r = \left(\frac{\delta\phi^r}{\delta\phi^r}\frac{\delta\mathcal{H}}{\delta\pi_r} - \frac{\delta\phi^r}{\delta\pi_r}\frac{\delta\mathcal{H}}{\delta\phi^r}\right)\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{24}$$

$$= \frac{\delta\mathcal{H}}{\delta\pi_r}\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{25}$$

$$\frac{d\pi^r}{dt} = \dot{\pi}^r = \left(\frac{\delta\pi^r}{\delta\phi^r}\frac{\delta\mathcal{H}}{\delta\pi_r} - \frac{\delta\pi^r}{\delta\pi_r}\frac{\delta\mathcal{H}}{\delta\phi^r}\right)\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{26}$$

$$= -\frac{\delta\mathcal{H}}{\delta\phi_r}\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{27}$$

I'm not too sure about the extra delta function that shows up here. Presumably it just serves to localize the result to one particular place, so that ϕ^r on the LHS and $\frac{\delta\mathcal{H}}{\delta\pi_r}$ must both be evaluated at the same location.

The commutators for quantum field theory are obtained by postulating the same relation between Poisson brackets and quantum commutators that we had in particle theory. Namely, we propose a commutator equivalent to each Poisson bracket (multiplied by $i\hbar$), so we have

$$\left[\phi^r,\pi_s\right] = i\hbar\delta_s^r\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{28}$$

$$\left[\phi^r,\phi^s\right] = 0 \tag{29}$$

$$\left[\pi_r,\pi_s\right] = 0 \tag{30}$$

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