FEYNMAN PROPAGATOR AS A CONTOUR INTEGRAL

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Section 3.13.2.

We've seen that we can write the Feynman propagator, defined as

$$i\Delta_F(x-y) \equiv \left\langle 0 \left| T\left[\phi(x)\phi^\dagger(y)\right] \right| 0 \right\rangle \tag{1}$$

in terms of commutators and that these commutators can be written as integrals over $\mathbf{k}$ space as

$$i\Delta^\pm(x-y) = \frac{1}{2(2\pi)^3}\int d^3\mathbf{k}\frac{e^{\mp ik(x-y)}}{\omega_\mathbf{k}} \tag{2}$$

We can convert these integrals into contour integrals, which turn out to simplify things in the long run. The conversion is a bit tricky, so it bears close scrutiny.

First, we introduce a complex variable k_0, which is the complex generalization of the energy component of the four-vector k. That is, the energy $\omega_\mathbf{k}$ is a point on the positive real axis in the complex plane where k_0 is the complex variable. (If you're used to doing complex variable stuff, the complex variable is usually denoted with the symbol z, so k_0 serves the role of z in what follows.) Now suppose we have some function $f(k_0)$ defined in this complex plane, and consider the contour integral

$$\int_{C^+}\frac{f(k_0)}{k_0-\omega_\mathbf{k}}dk_0 \tag{3}$$

where C^+ is any counterclockwise contour that encloses the point $k_0 = \omega_\mathbf{k}$. From Cauchy's residue theorem, the residue of the integrand is

$$\text{Res}\frac{f(k_0)}{k_0-\omega_\mathbf{k}} = \lim_{k_0\to\omega_\mathbf{k}}(k_0-\omega_\mathbf{k})\frac{f(k_0)}{k_0-\omega_\mathbf{k}} \tag{4}$$

$$= f(\omega_\mathbf{k}) \tag{5}$$

so the contour integral comes out to

$$\int_{C^+}\frac{f(k_0)}{k_0-\omega_{\mathbf{k}}}dk_0=2\pi i f(\omega_{\mathbf{k}}) \tag{6}$$

[This assumes that $f(k_0)$ has no poles within the contour.]

Returning to 2, we'll consider Δ^+ for now. The exponent inside the integrand is the scalar product of two four-vectors, so written out in full it is

$$-ik(x-y)=-i\omega_{\mathbf{k}}(t_x-t_y)+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) \tag{7}$$

Therefore, we can write the integral as

$$i\Delta^+(x-y)=\frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left[\frac{e^{-i\omega_{\mathbf{k}}(t_x-t_y)}}{2\omega_{\mathbf{k}}}\right]d^3\mathbf{k} \tag{8}$$

If we define a function as follows:

$$f(k_0)\equiv\frac{e^{-ik_0(t_x-t_y)}}{k_0+\omega_{\mathbf{k}}} \tag{9}$$

then the bracketed term in 8 is

$$f(\omega_{\mathbf{k}})=\frac{e^{-i\omega_{\mathbf{k}}(t_x-t_y)}}{2\omega_{\mathbf{k}}} \tag{10}$$

We can now use 6 to write $f(\omega_{\mathbf{k}})$ as a contour integral of the function $f(k_0)$, that is

$$f(\omega_{\mathbf{k}})=\frac{1}{2\pi i}\int_{C^+}\frac{f(k_0)}{k_0-\omega_{\mathbf{k}}}dk_0 \tag{11}$$

$$i\Delta^+(x-y)=\frac{1}{2\pi i}\frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left[\int_{C^+}\frac{e^{-ik_0(t_x-t_y)}}{(k_0+\omega_{\mathbf{k}})(k_0-\omega_{\mathbf{k}})}dk_0\right]d^3\mathbf{k} \tag{12}$$

At this point, Klauber writes the last line in the condensed form

$$i\Delta^+(x-y)=\frac{-i}{(2\pi)^4}\int_{C^+}\frac{e^{-ik(x-y)}}{k_0^2-\omega_{\mathbf{k}}^2}d^4k \tag{13}$$

This form can be confusing, since it consists of a contour integral around C^+ in the (complex) k_0 plane, and a spatial integral over all of the (real) 3-d space of $\mathbf{k}$. It's important to remember that the integral over k_0 is *only* around the contour C^+ and *not* over the entire k_0 plane, while the integrals over the components of $\mathbf{k}$ *are* over all *real* values of these components.

The denominator of 13 can now be transformed as follows, though I'm a bit unclear on why we can be sure this is allowed. First, we can write the four-vector scalar square modulus k^2 as

$$k^2 = k_0^2 - \mathbf{k}^2 \tag{14}$$

The reason this is a bit unclear is that k_0, as used here, is a *complex* variable while $\mathbf{k}$ is a real 3-vector. In any event, we can also use the relativistic energy-momentum relation

$$\omega_{\mathbf{k}}^2 - \mathbf{k}^2 = \mu^2 \tag{15}$$

and combine these two to get

$$k_0^2 - \omega_{\mathbf{k}}^2 = k^2 - \mu^2 \tag{16}$$

with the understanding that the k^2 on the RHS is a *complex* variable because it contains k_0^2. This allows us to rewrite 13 as

$$i\Delta^+(x-y) = \frac{-i}{(2\pi)^4}\int_{C^+}\frac{e^{-ik(x-y)}}{k^2-\mu^2}d^4k \tag{17}$$

The derivation for Δ^- is similar and given by Klauber in his section 3.17 (Appendix C to chapter 3). The result is

$$i\Delta^-(x-y) = \frac{i}{(2\pi)^4}\int_{C^-}\frac{e^{-ik(x-y)}}{k^2-\mu^2}d^4k \tag{18}$$

where the contour C^- is a counterclockwise contour around the point $k_0 = -\omega_{\mathbf{k}}$.