FEYNMAN PROPAGATOR AS A SINGLE REAL INTEGRAL

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Section 3.13.2.

We've seen that we can write the Feynman propagator, defined as

$$i\Delta_F(x-y)\equiv\left\langle 0\left|T\left[\phi(x)\phi^{\dagger}(y)\right]\right|0\right\rangle \tag{1}$$

in terms of commutators and that these commutators can be written as integrals over $\mathbf{k}$ space as

$$i\Delta^{\pm}(x-y)=\frac{1}{2(2\pi)^3}\int d^3\mathbf{k}\frac{e^{\mp ik(x-y)}}{\omega_{\mathbf{k}}} \tag{2}$$

These integrals over real 3-d $\mathbf{k}$ space can be converted to contour integrals, so that

$$i\Delta^{+}(x-y) = \frac{-i}{(2\pi)^4}\int_{C^+}\frac{e^{-ik(x-y)}}{k^2-\mu^2}d^4k \tag{3}$$

$$i\Delta^{-}(x-y) = \frac{i}{(2\pi)^4}\int_{C^-}\frac{e^{-ik(x-y)}}{k^2-\mu^2}d^4k \tag{4}$$

In these integrals, the integration is still done over real 3-d $\mathbf{k}$ space, but the energy component of the k four-vector has been generalized to a complex variable k_0, and it is the integral over k_0 that is a contour integral in each case. For Δ^+, the integral is over a counterclockwise contour C^+ which encloses the point $k_0=+\omega_{\mathbf{k}}$ but *excludes* the point $k_0=-\omega_{\mathbf{k}}$, while for Δ^-, the integral is over a different counterclockwise contour C^- which encloses the point $k_0=-\omega_{\mathbf{k}}$ and excludes the point $k_0=+\omega_{\mathbf{k}}$.From Cauchy's residue theorem, these contours $C^{\pm}$ can have any shape that satisfies the above conditions (that is, that they are counterclockwise and include only the points specified).

Since our goal is to convert the contour integrals back into integrals over k_0 along the real axis, we try specifying the contours $C^{\pm}$ as follows. Consider first the contour C^-. We can choose the contour to start at some large negative value on the real k_0 axis, then travel to the right until just before the point $k_0=-\omega_{\mathbf{k}}$ where it becomes a small semicircle that loops just below the point $k_0=-\omega_{\mathbf{k}}$, then continues along the real axis until just before the point $k_0 = +\omega_{\mathbf{k}}$, where it follows a small semicircle just *above* the point $k_0 = +\omega_{\mathbf{k}}$. After this, it extends out to some large positive value on the real axis, and then returns to its starting point by looping through a large semicircle in the upper half plane. [As these diagrams are hard to draw on a computer, I've tried to describe it in words, but if you draw it out yourself you should get the idea; you can alse see the diagram in Klauber's figure 3-6.] This contour satisfies all the conditions for C^- above, so if it is used in the integral 4, it will give the desired result.

Now, for Δ^+, note by comparing 3 and 4 that the integrands in both cases are the same; the only differences are that Δ^+ has a minus sign out front and is integrated over a different contour (C^+ instead of C^-). To define C^+, we start with the same path along the real k_0 axis as we did for C^-, except this time we close the curve by drawing a large semicircle in the lower half plane from the point on the right of the real axis back to the starting point on the left of the real axis. This curve encloses $k_0 = +\omega_{\mathbf{k}}$ and excludes $k_0 = -\omega_{\mathbf{k}}$, but it is traversed in a *clockwise* direction, which results in it giving the negative of the integral that would be obtained by going round the curve in the usual counterclockwise direction, so this contour is actually C^+ in reverse. However, this is exactly what we want, since the integral in 3 has a minus sign out front. We can therefore write $\Delta^+$and Δ^- using the *same* integral; it is only the contour that is different. Since the Feynman propagator is written as

$$i\Delta_F(x-y) = \left\langle 0 \left| i\Delta^{\pm}(x-y) \right| 0 \right\rangle \tag{5}$$

we can therefore write it as

$$i\Delta_F(x-y) = \frac{i}{(2\pi)^4} \int_{C_F} \frac{e^{-ik(x-y)}}{k^2-\mu^2} d^4k \tag{6}$$

where the contour C_F is one of the two contours defined above. [Calling the contours by the single symbol C_F can be a bit confusing, since C_F actually refers to two different contours depending on whether we're considering a particle or an antiparticle. However, as we'll see, both contours collapse into a single integration path, so hopefully the confusion will disappear when that happens.]

If you've done any contour integration before, it won't surprise you to discover that our goal now is to let the left and right endpoints of the contours on the real axis tend to infinity and show that the integral over the large semicircular portion joining these endpoints tends to zero as we do so. When we do this, the integrals over the two contours become equal, since the only place where the two contours differed was in these large semicircular loops. What we're left with is an integral from $-\infty$ to $+\infty$ along the real k_0 axis (with the two little loops around the points $k_0 = \pm\omega_{\mathbf{k}}$).

The proof that the integrals over these large semicircles tend to zero is similar to the method used in Example 2 in the earlier post on contour integration. Looking at the upper semicircle first, on that semicircle we can parameterize k_0 with a variable w:

$$\begin{aligned} w &\in [0,\pi] && (7)\\ k_0 &= Re^{iw} && (8)\\ dk_0 &= iRe^{iw}dw && (9) \end{aligned}$$

where R is the radius of the semicircle. The integral of k_0 over the semicircle is then (we can ignore the factors that don't depend on k_0 here; also since we're interested in letting $R\to\infty$ we can approximate the denominator by $k^2-\mu^2 \to k_0^2 = R^2e^{2iw}$):

$$\int_0^{\pi}\frac{e^{-ik_0\left(t_x-t_y\right)}}{R^2e^{2iw}}iRe^{iw}dw = i\int_0^{\pi}\frac{e^{-iRe^{iw}\left(t_x-t_y\right)}}{Re^{iw}}dw \qquad (10)$$

The exponent in the numerator is

$$-iRe^{iw}\left(t_x-t_y\right) = R\left(t_x-t_y\right)\left[-i\cos w+\sin w\right] \qquad (11)$$

Since this contour corresponds to the antiparticle case, then $t_x < t_y$ and since $w\in[0,\pi]$, $\sin w > 0$, so the real part of this exponent is always negative and tends to $-\infty$ as $R\to\infty$. Thus the integrand in 10 tends to zero for large R so the integral over this part of the contour tends to zero.

For the lower semicircle, the same parameterization gives

$$\begin{aligned} w &\in [0,-\pi] && (12)\\ k_0 &= Re^{iw} && (13)\\ dk_0 &= iRe^{iw}dw && (14) \end{aligned}$$

This contour is for the particle case where $t_x > t_y$ and since $w\in[0,-\pi]$, $\sin w < 0$ so again the exponent is negative and tends to $-\infty$ as $R\to\infty$, so again the integral tends to zero. Thus both integrals reduce to the *same* integral along the real k_0 axis from $-\infty$ to $+\infty$ (with the two little semicircular arcs around the points $k_0 = \pm\omega_{\mathbf{k}}$).

To deal with these little semicircles, a fudge is applied. The idea is to shift the poles slightly away from the real axis. The denominator in 6 is equivalent to

$$k^2 - \mu^2 = k_0^2 - \omega_{\mathbf{k}}^2 \tag{15}$$

so we shift the poles from $k_0 = \pm\omega_{\mathbf{k}}$ to $k_0 = \pm(\omega_{\mathbf{k}} - i\eta)$ where η is some small real quantity. This shifts the pole at $k_0 = +\omega_{\mathbf{k}}$ to $\omega_{\mathbf{k}} - i\eta$ so it's slightly below the real axis, and the pole at $k_0 = -\omega_{\mathbf{k}}$ to $-\omega_{\mathbf{k}} + i\eta$ so it's slightly above the real axis. If we now change the path of integration to be along the real axis all the way from $-\infty$ to $+\infty$ (with no bumps along the way), then the shifted poles are still on the same sides of the contour that they were before. The idea is then that we can do the integral along the real axis and *then* take the limit as $\eta \to 0$:

$$i\Delta_F(x-y) = \lim_{\eta\to 0} \frac{i}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{e^{-ik(x-y)}}{k_0^2 - (\omega_{\mathbf{k}} - i\eta)^2} d^4k \tag{16}$$

Now the infinite limts apply to all four components of the four-vector k, and there is no contour integration any more.

In practice, we can define $\varepsilon \equiv 2\eta\,\omega_{\mathbf{k}}$ and keep only the first order term in ε, so that

$$k_0^2 - (\omega_{\mathbf{k}} - i\eta)^2 \cong k_0^2 - \omega_{\mathbf{k}}^2 + i\varepsilon = k^2 - \mu^2 + i\varepsilon \tag{17}$$

The Feynman propagator then becomes

$$i\Delta_F(x-y) = \lim_{\varepsilon\to 0} \frac{i}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{e^{-ik(x-y)}}{k^2 - \mu^2 + i\varepsilon} d^4k \tag{18}$$

In this integral, all the components of k are now real, and there is no contour integration. [I'm assuming that all this effort makes more sense when interactions are considered, since as it stands, it seems that the integral consists of just a complex exponential over a polynomial, which I assume could be done. It's not clear how the $i\varepsilon$ plays a role in all this. Oh well, I'll be patient and see what develops.]