

DIRAC SPIN OPERATOR IN QUANTUM FIELD THEORY

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4.

The total spin of a multiparticle Dirac state is a bit trickier to calculate than the total momentum. For the momentum, the result turned out to be

$$\mathbf{P} = \sum_{r, \mathbf{p}} \mathbf{p} (N_r(\mathbf{p}) + \bar{N}_r(\mathbf{p})) \quad (1)$$

which is just the sum of the momenta of the particles and antiparticles. This works because a multiparticle state is an eigenstate of all the number operators, with the eigenvalues being just the number of particles in each state with spin r and momentum \mathbf{p} , and \mathbf{p} itself is just a 3-vector which multiplies the result. For example, we can operate on a multiparticle state to get the total momentum like this:

$$\mathbf{P} |\psi_{r_1 \mathbf{p}_1} \psi_{r_2 \mathbf{p}_2} \psi_{r_3 \mathbf{p}_3} \bar{\psi}_{r_1 \mathbf{p}_1}\rangle = 2\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \quad (2)$$

If we tried something analogous for the spin component Σ_j we would get

$$\Sigma_{j, tot} = \sum_{r, \mathbf{p}} \Sigma_j (N_r(\mathbf{p}) + \bar{N}_r(\mathbf{p})) \quad (3)$$

As with the momentum, a multiparticle state is still an eigenstate of the number operators, but Σ_j is a matrix operator which can operate only on single particle states, that is, states containing a single 4-component spinor. That is, the operation

$$\Sigma_j |\psi_{r_1 \mathbf{p}_1} \psi_{r_2 \mathbf{p}_2} \psi_{r_3 \mathbf{p}_3} \bar{\psi}_{r_1 \mathbf{p}_1}\rangle \quad (4)$$

is not defined, so Σ_j is not a well-defined quantity.

The solution turns out to be defining the total spin operator as

$${}_{QFT} \Sigma_j = \int_V \psi^\dagger \Sigma_j \psi d^3x \quad (5)$$

where the Σ_j in the integrand is the usual Dirac spin operator, and ψ and ψ^\dagger are the general solutions to the Dirac equation

$$\psi = \sum_{r=1}^2 \sum_{\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} \left[c_r(\mathbf{p}) u_r(\mathbf{p}) e^{-ipx} + d_r^\dagger(\mathbf{p}) v_r(\mathbf{p}) e^{ipx} \right] \quad (6)$$

$$\equiv \psi^+ + \psi^- \quad (7)$$

$$\psi^\dagger = \sum_{r=1}^2 \sum_{\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} \left[d_r(\mathbf{p}) v_r^\dagger(\mathbf{p}) e^{-ipx} + c_r^\dagger(\mathbf{p}) u_r^\dagger(\mathbf{p}) e^{ipx} \right] \quad (8)$$

$$\equiv \psi^{\dagger+} + \psi^{\dagger-} \quad (9)$$

Klauber evaluates the integral in his section 4.9.1 for the case of ψ and ψ^\dagger containing only c and c^\dagger operators. The integration uses the usual property of such integrals that any term in the integrand containing an exponential e^{ipx} goes to zero because of the boundary conditions. We are left with

$$Q_{FT} \Sigma_j = \sum_{r,s,\mathbf{p}} \frac{m}{E_{\mathbf{p}}} u_r^\dagger(\mathbf{p}) \Sigma_j u_s(\mathbf{p}) c_r^\dagger(\mathbf{p}) c_s(\mathbf{p}) \quad (10)$$

If we apply this operator to some state $|\psi_{s'\mathbf{p}'}\rangle$ then because the $c_s(\mathbf{p})$ operator (an annihilation operator) is the first one to operate on the state, only operators with $s = s'$ and $\mathbf{p} = \mathbf{p}'$ will produce a non-zero result. In those cases, the $s'\mathbf{p}'$ particle is annihilated and then replaced with a $r\mathbf{p}$ particle because of the creation operator $c_r^\dagger(\mathbf{p})$. That is, the sum over \mathbf{p} collapses to a single term where $\mathbf{p} = \mathbf{p}'$ and the sum over s is eliminated:

$$Q_{FT} \Sigma_j |\psi_{s'\mathbf{p}'}\rangle = \sum_r \frac{m}{E_{\mathbf{p}'}} u_r^\dagger(\mathbf{p}') \Sigma_j u_r(\mathbf{p}') |\psi_{s'\mathbf{p}'}\rangle \quad (11)$$

Now suppose we apply this operator to a 2-particle state $|\psi_{s'\mathbf{p}'}\psi_{s''\mathbf{p}''}\rangle$. We'll get a term like 11 for each particle, with the result

$$Q_{FT} \Sigma_j |\psi_{s'\mathbf{p}'}\psi_{s''\mathbf{p}''}\rangle = \left[\sum_r \frac{m}{E_{\mathbf{p}'}} u_r^\dagger(\mathbf{p}') \Sigma_j u_r(\mathbf{p}') + \sum_r \frac{m}{E_{\mathbf{p}''}} u_r^\dagger(\mathbf{p}'') \Sigma_j u_r(\mathbf{p}'') \right] |\psi_{s'\mathbf{p}'}\psi_{s''\mathbf{p}''}\rangle \quad (12)$$

Thus in general, because only the terms in the sum over \mathbf{p} in 10 that correspond to the momenta of the particles in the many particle state will be non-zero when this operator is applied to that many particle state, and the sum over s also collapses, we can write 10 as

$$Q_{FT} \Sigma_j = \sum_{r,\mathbf{p}} \frac{m}{E_{\mathbf{p}}} u_r^\dagger(\mathbf{p}) \Sigma_j u_r(\mathbf{p}) c_r^\dagger(\mathbf{p}) c_r(\mathbf{p}) \quad (13)$$

$$= \sum_{r,\mathbf{p}} \frac{m}{E_{\mathbf{p}}} u_r^\dagger(\mathbf{p}) \Sigma_j u_r(\mathbf{p}) N_r(\mathbf{p}) \quad (14)$$

[Klauber's equation 4-113 is a bit sloppy since he applies 10 to a state which he calls $|\psi_{s\mathbf{p}}\rangle$ which contains the two summation variables s and \mathbf{p} , when in fact this state should refer to a specific spin and momentum and not be part of the sum over s and \mathbf{p} . Likewise, he retains the sum over \mathbf{p} in 4-114 even though the annihilation operator removes all but one momentum. The final result 4-115 does appear to be correct however.]

The expectation value of the operator $Q_{FT}\Sigma_j$ between two multiparticle states $|\psi_1\rangle$ and $|\psi_2\rangle$ is therefore

$$\langle \psi_1 | Q_{FT}\Sigma_j | \psi_2 \rangle \quad (15)$$

Because the term $u_r^\dagger(\mathbf{p})\Sigma_j u_r(\mathbf{p})$ is the product of a 1×4 row vector ($u_r^\dagger(\mathbf{p})$), a 4×4 matrix (Σ_j) and a 4×1 column vector ($u_r(\mathbf{p})$), the result is just a scalar, that is, a number. Also, because the u_r spinors are eigenspinors of the Σ_3 operator, we have $\Sigma_3 u_{up} = +\frac{1}{2}u_{up}$ and $\Sigma_3 u_{down} = -\frac{1}{2}u_{down}$. Finally, the inner product $u_r^\dagger(\mathbf{p})u_r(\mathbf{p}) = E_{\mathbf{p}}/m$. So applying 14 to the state $|\psi_{up,\mathbf{p}'}\psi_{up,\mathbf{p}''}\rangle$, for example, and calculating the expectation value in that state, we get

$$\begin{aligned} \langle \psi_{up,\mathbf{p}'}\psi_{up,\mathbf{p}''} | Q_{FT}\Sigma_3 | \psi_{up,\mathbf{p}'}\psi_{up,\mathbf{p}''} \rangle &= \left\langle \psi_{up,\mathbf{p}'}\psi_{up,\mathbf{p}''} \left| \left[+\frac{1}{2} + \frac{1}{2} \right] \right| \psi_{up,\mathbf{p}'}\psi_{up,\mathbf{p}''} \right\rangle \\ &= +1 \langle \psi_{up,\mathbf{p}'}\psi_{up,\mathbf{p}''} \rangle \end{aligned} \quad (17)$$

The expectation value of $Q_{FT}\Sigma_j$ between any two different states produces zero because different multiparticle states are orthogonal.