DIRAC SPIN OPERATOR IN QUANTUM FIELD THEORY

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4.

The total spin of a multiparticle Dirac state is a bit trickier to calculate than the total momentum. For the momentum, the result turned out to be

$$\mathbf{P}=\sum_{r,\mathbf{p}}\mathbf{p}\left(N_{r}\left(\mathbf{p}\right)+\bar{N}_{r}\left(\mathbf{p}\right)\right) \tag{1}$$

which is just the sum of the momenta of the particles and antiparticles. This works because a multiparticle state is an eigenstate of all the number operators, with the eigenvalues being just the number of particles in each state with spin r and momentum $\mathbf{p}$, and $\mathbf{p}$ itself is just a 3-vector which multiplies the result. For example, we can operate on a multiparticle state to get the total momentum like this:

$$\mathbf{P}\left|\psi_{r_1\mathbf{p}_1}\psi_{r_2\mathbf{p}_2}\psi_{r_1\mathbf{p}_3}\bar{\psi}_{r_1\mathbf{p}_1}\right\rangle=2\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3 \tag{2}$$

If we tried something analogous for the spin componentΣ_j we would get

$$\Sigma_{j,tot}=\sum_{r,\mathbf{p}}\Sigma_j\left(N_{r}\left(\mathbf{p}\right)+\bar{N}_{r}\left(\mathbf{p}\right)\right) \tag{3}$$

As with the momentum, a multiparticle state is still an eigenstate of the number operators, but Σ_j is a matrix operator which can operate only on single particle states, that is, states containing a single 4-component spinor. That is, the operation

$$\Sigma_j\left|\psi_{r_1\mathbf{p}_1}\psi_{r_2\mathbf{p}_2}\psi_{r_1\mathbf{p}_3}\bar{\psi}_{r_1\mathbf{p}_1}\right\rangle \tag{4}$$

is not defined, so 3 is not a well-defined quantity.

The solution turns out to be defining the total spin operator as

$${}_{QFT}\Sigma_j=\int_V\psi^{\dagger}\Sigma_j\psi\,d^3x \tag{5}$$

where the Σ_j in the integrand is the usual Dirac spin operator, and ψ and $\psi^{\dagger}$ are the general solutionsto the Dirac equation

$$\psi = \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[c_r(\mathbf{p})\,u_r(\mathbf{p})\,e^{-ipx}+d_r^{\dagger}(\mathbf{p})\,v_r(\mathbf{p})\,e^{ipx}\right] \tag{6}$$

$$\equiv \psi^{+}+\psi^{-} \tag{7}$$

$$\psi^{\dagger} = \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[d_r(\mathbf{p})\,v_r^{\dagger}(\mathbf{p})\,e^{-ipx}+c_r^{\dagger}(\mathbf{p})\,u_r^{\dagger}(\mathbf{p})\,e^{ipx}\right] \tag{8}$$

$$\equiv \psi^{\dagger+}+\psi^{\dagger-} \tag{9}$$

Klauber evaluates the integral in his section 4.9.1 for the case of ψ and $\psi^{\dagger}$ containing only c and $c^{\dagger}$ operators. The integration uses the usual property of such integrals that any term in the integrand containing an exponential e^{ipx} goes to zero because of the boundary conditions. We are left with

$$_{QFT}\Sigma_j = \sum_{r,s,\mathbf{p}}\frac{m}{E_{\mathbf{p}}}u_r^{\dagger}(\mathbf{p})\,\Sigma_j u_s(\mathbf{p})\,c_r^{\dagger}(\mathbf{p})\,c_s(\mathbf{p}) \tag{10}$$

If we apply this operator to some state $\left|\psi_{s'\mathbf{p}'}\right\rangle$ then because the $c_s(\mathbf{p})$ operator (an annihiliation operator) is the first one to operate on the state, only operators with $s=s'$ and $\mathbf{p}=\mathbf{p}'$ will produce a non-zero result. In those cases, the $s'\mathbf{p}'$ particle is annihilated and then replaced with a $r\mathbf{p}$ particle because of the creation operator $c_r^{\dagger}(\mathbf{p})$. That is, the sum over $\mathbf{p}$ collapses to a single term where $\mathbf{p}=\mathbf{p}'$ and the sum over s is eliminated:

$$_{QFT}\Sigma_j\left|\psi_{s'\mathbf{p}'}\right\rangle = \sum_{r}\frac{m}{E_{\mathbf{p}'}}u_r^{\dagger}(\mathbf{p}')\,\Sigma_j u_r(\mathbf{p}')\left|\psi_{s'\mathbf{p}'}\right\rangle \tag{11}$$

Now suppose we apply this operator to a 2-particle state $\left|\psi_{s'\mathbf{p}'}\psi_{s''\mathbf{p}''}\right\rangle$. We'll get a term like 11 for each particle, with the result

$$_{QFT}\Sigma_j\left|\psi_{s'\mathbf{p}'}\psi_{s''\mathbf{p}''}\right\rangle = \left[\sum_{r}\frac{m}{E_{\mathbf{p}'}}u_r^{\dagger}(\mathbf{p}')\,\Sigma_j u_r(\mathbf{p}')+\sum_{r}\frac{m}{E_{\mathbf{p}''}}u_r^{\dagger}(\mathbf{p}'')\,\Sigma_j u_r(\mathbf{p}'')\right]\left|\psi_{s'\mathbf{p}'}\psi_{s''\mathbf{p}''}\right\rangle \tag{12}$$

Thus in general, because only the terms in the sum over $\mathbf{p}$ in 10 that correspond to the momenta of the particles in the many particle state will be non-zero when this operator is applied to that many particle state, and the sum over s also collapses, we can write 10 as

$$_{QFT}\Sigma_j = \sum_{r,\mathbf{p}}\frac{m}{E_{\mathbf{p}}}u_r^{\dagger}(\mathbf{p})\,\Sigma_j u_r(\mathbf{p})\,c_r^{\dagger}(\mathbf{p})\,c_r(\mathbf{p}) \tag{13}$$

$$= \sum_{r,\mathbf{p}}\frac{m}{E_{\mathbf{p}}}u_r^{\dagger}(\mathbf{p})\,\Sigma_j u_r(\mathbf{p})\,N_r(\mathbf{p}) \tag{14}$$

[Klauber's equation 4-113 is a bit sloppy since he applies 10 to a state which he calls $\left|\psi_{s\mathbf{p}}\right\rangle$ which contains the two summation variables s and $\mathbf{p}$, when in fact this state should refer to a specific spin and momentum and not be part of the sum over s and $\mathbf{p}$. Likewise, he retains the sum over $\mathbf{p}$ in 4-114 even though the annihilation operator removes all but one momentum. The final result 4-115 does appear to be correct however.]

The expectation value of the operator ${}_{QFT}\Sigma_j$ between two multiparticle states $|\psi_1\rangle$ and $|\psi_2\rangle$ is therefore

$$\left\langle \psi_1 \left|{}_{QFT}\Sigma_j\right| \psi_2 \right\rangle \tag{15}$$

Because the term $u_r^\dagger(\mathbf{p})\,\Sigma_j u_r(\mathbf{p})$ is the product of a 1×4 row vector $(u_r^\dagger(\mathbf{p}))$, a 4×4 matrix (Σ_j) and a 4×1 column vector $(u_r(\mathbf{p}))$, the result is just a scalar, that is, a number. Also, because the u_r spinors are eigenspinors of the Σ_3 operator, we have $\Sigma_3 u_{up} = +\frac{1}{2}u_{up}$ and $\Sigma_3 u_{down} = -\frac{1}{2}u_{down}$. Finally, the inner product$u_r^\dagger(\mathbf{p})\,u_r(\mathbf{p}) = E_{\mathbf{p}}/m$. So applying 14 to the state $\left|\psi_{up,\mathbf{p}'}\psi_{up,\mathbf{p}''}\right\rangle$, for example, and calculating the expectation value in that state, we get

$$\begin{aligned}
\left\langle \psi_{up,\mathbf{p}'}\psi_{up,\mathbf{p}''} \left|{}_{QFT}\Sigma_3\right| \psi_{up,\mathbf{p}'}\psi_{up,\mathbf{p}''} \right\rangle &= \left\langle \psi_{up,\mathbf{p}'}\psi_{up,\mathbf{p}''} \left| \left[+\frac{1}{2}+\frac{1}{2}\right] \right| \psi_{up,\mathbf{p}'}\psi_{up,\mathbf{p}''} \right\rangle &(16)\\
&= +1\left|\psi_{up,\mathbf{p}'}\psi_{up,\mathbf{p}''}\right\rangle &(17)
\end{aligned}$$

The expectation value of ${}_{QFT}\Sigma_j$ between any two different states produces zero because different multiparticle states are orthogonal.