NATURAL UNITS

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References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 2, Problems 2.1-2.2.

In quantum field theory, the quantities c (speed of light) and $\hbar$ (Planck's constant h divided by 2π) occur frequently. Both of these quantities are (or at least are believed to be) absolute fundamental constants of nature, so expressing their values in terms of arbitrary units such as those in the MKS or CGS systems seems rather artificial and unnecessary. It is common practice in QFT, therefore, to take both c and $\hbar$ as basic units by which everything else is measured. As such, we set $c=1$ (no units) and $\hbar=1$ (also no units). We've already seen that taking $c=1$ is common practice when using relativity theory, but there we weren't concerned with quantum mechanics so no mention was made of $\hbar$. Here we'll explore the consequences of setting $c=\hbar=1$, a system known as *natural units*.

To work out what this choice means, we need to relate these units to those with which we're more familiar. Klauber treats the CGS system, but since I've used MKS for most of my posts, we'll relate things back to that instead.

In MKS, c is a velocity so it has dimensions of length $\div$ time. Making c dimensionless means that all velocities are dimensionless, so that the unit of length is also the unit of time.

What about $\hbar$? In MKS, its units are those of action, or energy $\times$ time $=$ mass $\times$ $(\text{length})^2 \div$ time. In natural units, the units of length are time are the same, so $\hbar$ has units of mass $\times$ length. Making this dimensionless means that the dimension of length (and thus also time) is the inverse of the dimension of mass. We've thus managed to reduce the three distinct units (length, mass and time) in the MKS system to a single unit (mass). We therefore need some basic unit of mass. There are various ways we could choose the mass unit, but the most commonly used unit in QFT is the MeV (mega-electron-volt), which is the energy an electron gains by being accelerated through a potential difference of 10^6 volts. This isn't technically a 'natural' unit, since although the energy *is* expressed in terms of a fundamental constant (the charge on the electron), it also uses a unit (the volt) that is derived from the MKS system of units. However, it's what's in common use.

From these definitions, it's possible to work out the units of any physical quantity entirely in terms of powers of mass. Klauber's Wholeness Chart 2-1 shows many of these quantities. Energy, mass and acceleration all have units of mass, while length and time have units of inverse mass. Area has units of $(\text{mass})^{-2}$ and volume of $(\text{mass})^{-3}$.

To convert from natural units to so-called *hybrid units*, in which length and time have units taken from either the MKS or CGS systems, but mass is still given in terms of MeV, we first write out c and $\hbar$ in hybrid units (here I'm using MKS):

$$c = 2.99 \times 10^{8} \text{ m s}^{-1} \tag{1}$$

$$\hbar = 6.58 \times 10^{-22} \text{ MeV s} \tag{2}$$

$$\hbar c = 1.973 \times 10^{-13} \text{ MeV m} \tag{3}$$

Next, we multiply the quantity in natural units by factors of c and/or $\hbar$ to make the units come out right in the hybrid system. Note that in the hybrid system, energy (in MeV) is still a fundamental unit, so that mass is expressed in units of energy $\div c^2 = \text{MeV s}^2\text{m}^{-2}$.

For example, in natural units, length has dimensions of MeV^{-1}, so to get a length in metres, we multiply it by $\hbar c$. Time also has dimensions of MeV^{-1} so to get a time in seconds, multiply it by $\hbar$. Force has units of energy $\div$ length and energy is the same in natural and hybrid units, so to get force as MeV m^{-1} we divide it $\hbar c$. And so on.

To make the final conversion to MKS units, we need the conversion

$$1 \text{ MeV} = 1.60218 \times 10^{-13} \text{ J} \tag{4}$$

Thus any hybrid quantity containing a power of MeV gets multiplied by the same power of this conversion factor to get the final result in MKS.

We can in fact define a system of units based on any appropriate set of physical constants that we like. The MKS system uses the metre (ultimately based on the size of the Earth; one early definition was that 1 metre is 10^{-7} times the distance from the north pole to the equator), the kilogram (the weight of 1 litre of water at 4° C, which might sound fundamental, but the litre, of course, is defined from the metre, so again, this unit depends on properties of the Earth), and the second (a unit of time ultimately based on the Earth's rotation period). From the point of view of fundamental physics, all three of these units are arbitrary as none of them are based on any fundamental constants of nature.

We could, for example, define a system of units in which one of the 'fundamental' units is the speed of sound. We would need to define precisely *how* the speed of sound is to be measured, however, since it depends on the substance transmitting the sound. In general, sound travels faster through denser materials. Suppose we define the material in which the speed is to be measured (and its temperature and pressure), and set this speed to be

$s = 1$ (dimensionless). As with $c = 1$ above, setting a velocity to be a dimensionless quantity implies that length and time have the same units. If we retained the second as the unit of time, then length is also measured in seconds.

As another example, the fine structure constant that turns up in the spectrum of hydrogen is (in MKS):

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137.036} \tag{5}$$

To convert this to natural units, we need to know how to handle electric charge. There are two commonly used ways of doing this. In strict CGS or Gaussian units, the factor $4\pi\varepsilon_0$ is defined to be 1 and dimensionless, which makes Coulomb's law take the form

$$F = \frac{q_1 q_2}{r^2} \tag{6}$$

for the force F between two charges separated by a distance r. This means that the units of charge can actually be expressed in terms of length, mass and time, since

$$\frac{\text{mass} \times \text{length}}{(\text{time})^2} = \frac{(\text{charge})^2}{(\text{length})^2} \tag{7}$$

$$\text{charge} = \sqrt{\frac{\text{mass} \times (\text{length})^3}{(\text{time})^2}} \tag{8}$$

However, another system defines just ε_0 on its own (without the 4π) to be 1 and dimensionless. The units of charge come out the same in terms of mass, length and time, but the numerical values are, of course, different. This latter system is the more common in QFT, so in those units

$$\alpha = \frac{e^2}{4\pi\hbar c} = \frac{1}{137.036} \tag{9}$$

In natural units, this becomes

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137.036} \tag{10}$$

which gives a value for the electron charge of

$$e = \sqrt{\frac{4\pi}{137.036}} = 0.3028 \text{ (dimensionless)} \tag{11}$$

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