**KLEIN-GORDON EQUATION: PLANE WAVE SOLUTIONS**

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.1.

The Klein-Gordon equation was one of the first attempts at producing a relativistic quantum theory. In natural units, the equation is

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi = 0 \tag{1}$$

This equation also results from the Euler-Lagrange equation for a scalar field $\phi$with Lagrangian

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2}m^2\phi^2 \tag{2}$$

This is the Lagrangian for zero potential $V(\phi) = 0$.

To write solutions to equation 1, we can introduce some new notation. In natural units, the four-momentum is

$$\begin{aligned} p_\mu &= \left[\begin{array}{cc} E & p_i \end{array}\right] & (3)\\ &= \left[\begin{array}{cc} E & -p^i \end{array}\right] & (4)\end{aligned}$$

The scalar product of four-momentum with a spacetime vector $x^\mu$ is therefore

$$px \equiv p_\mu x^\mu = Et - \mathbf{p}\cdot\mathbf{x} \tag{5}$$

For a plane wave with angular frequency $\omega$, Planck's relation is $E = \hbar\omega = \omega$, and the wave vector $\mathbf{k}$ has components in the three spatial directions of $2\pi/\lambda_i$, where $\lambda_i$ is the component of the wavelength in direction $x_i$. For example, a wave moving in the $x_1$ direction has $\lambda_2 = \lambda_3 = \infty$, so $\mathbf{k} = [k_1, 0, 0]$. The four-vector $k^\mu$ is

$$k^\mu = [\omega, \mathbf{k}] \tag{6}$$

and since $\mathbf{p} = \mathbf{k}$ in natural units, we have

$$kx = k_\mu x^\mu = p_\mu x^\mu = px \tag{7}$$

A plane wave solution to 1 turns out to be

$$\phi=\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(A_{\mathbf{k}}e^{-ikx}+B_{\mathbf{k}}^{\dagger}e^{ikx}\right) \tag{8}$$

We can see this by direct substitution. Consider one term $\phi_{\mathbf{k}}$ from the sum. Then

$$\phi_{\mathbf{k}}=\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(A_{\mathbf{k}}e^{-ikx}+B_{\mathbf{k}}^{\dagger}e^{ikx}\right) \tag{9}$$

$$\partial^{\mu}\phi_{\mathbf{k}}=\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(-ik_{\mu}A_{\mathbf{k}}e^{-ikx}+ik_{\mu}B_{\mathbf{k}}^{\dagger}e^{ikx}\right) \tag{10}$$

$$\partial_{\mu}\partial^{\mu}\phi_{\mathbf{k}}=\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(-k_{\mu}k^{\mu}A_{\mathbf{k}}e^{-ikx}-k_{\mu}k^{\mu}B_{\mathbf{k}}^{\dagger}e^{ikx}\right) \tag{11}$$

However, using the invariant scalar $p_{\mu}p^{\mu}$ from relativity:

$$k_{\mu}k^{\mu}=p_{\mu}p^{\mu}=E^2-p^2=m^2 \tag{12}$$

Thus

$$\partial_{\mu}\partial^{\mu}\phi_{\mathbf{k}}=-m^2\phi_{\mathbf{k}} \tag{13}$$

so 1 is true for a single component $\phi_{\mathbf{k}}$. Since the solution 8 is a linear combination of such solutions, and the original differential equation is linear, then 8 is also a solution. [The normalization factor $1/\sqrt{2V\omega_{\mathbf{k}}}$ is irrelevant in proving that 8 is a solution; it's just there to make future calculations easier.]

Note that the first term (involving $A_{\mathbf{k}}$) is also a solution of the free-particle Schrödinger equation, which is

$$i\frac{\partial\phi_S}{\partial t}=-\frac{1}{2m}\nabla^2\phi_S \tag{14}$$

If we take

$$\phi_S=\sum_{\mathbf{k}}A_{\mathbf{k}}e^{-ikx} \tag{15}$$

then

$$i\frac{\partial\phi_S}{\partial t} = \sum_{\mathbf{k}} E_{\mathbf{k}} A_{\mathbf{k}} e^{-ikx} \tag{16}$$

$$-\frac{1}{2m}\nabla^2\phi_S = \sum_{\mathbf{k}} \frac{k^2}{2m} A_{\mathbf{k}} e^{-ikx} \tag{17}$$

For a free particle, the energy $E_{\mathbf{k}} = \frac{p^2}{2m} = \frac{k^2}{2m}$, so the Schrödinger equation is satisfied. The second term (with $B_{\mathbf{k}}^{\dagger}$) does *not* satisfy the Schrödinger equation, since in that case we get

$$\phi_S = \sum_{\mathbf{k}} B_{\mathbf{k}}^{\dagger} e^{ikx} \tag{18}$$

$$i\frac{\partial\phi_S}{\partial t} = -\sum_{\mathbf{k}} E_{\mathbf{k}} B_{\mathbf{k}}^{\dagger} e^{ikx} \tag{19}$$

$$-\frac{1}{2m}\nabla^2\phi_S = \sum_{\mathbf{k}} \frac{k^2}{2m} B_{\mathbf{k}}^{\dagger} e^{ikx} \tag{20}$$

The extra minus sign means the two sides don't match.

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