KLEIN-GORDON EQUATION: ORTHONORMALITY OF SOLUTIONS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.2.

The plane wave solutions of the Klein-Gordon equation are

$$\phi=\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(A_{\mathbf{k}}e^{-ikx}+B_{\mathbf{k}}^{\dagger}e^{ikx}\right) \tag{1}$$

We can redefine a couple of terms by introducing

$$\phi_{\mathbf{k},A} \equiv \frac{e^{-ikx}}{\sqrt{V}} \tag{2}$$

$$\phi_{\mathbf{k},B^{\dagger}} \equiv \frac{e^{ikx}}{\sqrt{V}} \tag{3}$$

Then

$$\phi=\sum_{\mathbf{k}}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(A_{\mathbf{k}}\phi_{\mathbf{k},A}+B_{\mathbf{k}}^{\dagger}\phi_{\mathbf{k},B^{\dagger}}\right) \tag{4}$$

The $\phi_{\mathbf{k},A}$ and $\phi_{\mathbf{k},B^{\dagger}}$ are orthonormal functions. We have

$$\int\phi_{\mathbf{k},A}^{\dagger}\phi_{\mathbf{k}',A}d^{3}x=\frac{1}{V}\int e^{i(k'-k)x}d^{3}x \tag{5}$$

where the integral is over the volume V, and the wavelengths of the plane waves fit an integral number of times within V, so that the amplitudes of the waves at the boundaries are all zero. The four-vector k is defined as

$$k=[\omega_{\mathbf{k}},\mathbf{k}] \tag{6}$$

If $k'=k$, the integrand is 1 and is integrated over V, so the result is

$$\int \phi_{\mathbf{k},A}^{\dagger}\phi_{\mathbf{k},A}d^{3}x = \frac{1}{V}\int e^{i(k'-k)x}d^{3}x \tag{7}$$

$$= \frac{V}{V} = 1 \tag{8}$$

If $k' \neq k$, consider the integral over $x^1 = x$ (for the purposes of this derivation only, x refers to the single x dimension of the 3-vector $\mathbf{x}$ and should not be confused with the four-vector x used in 1):

$$\int \phi_{\mathbf{k}',A}^{\dagger}\phi_{\mathbf{k},A}dx = \frac{1}{V}e^{i\left(\omega_{\mathbf{k}'}-\omega_{\mathbf{k}}\right)t}e^{-i\left(k_{y}-k'_{y}\right)y}e^{-i\left(k_{z}-k'_{z}\right)z}\int e^{-i(k_{x}-k'_{x})x}dx \tag{9}$$

$$= -\frac{1}{i\left(k_{x}-k'_{x}\right)V}e^{i\left(\omega_{\mathbf{k}'}-\omega_{\mathbf{k}}\right)t}e^{-i\left(k_{y}-k'_{y}\right)y}e^{-i\left(k_{z}-k'_{z}\right)z}\left[e^{-i(k_{x}-k'_{x})x}\right]_{x=x_{0}}^{x=x_{1}} \tag{10}$$

$$= 0 \tag{11}$$

where x_0 and x_1 are the x limits of V, where by assumption the wave amplitude is zero. Therefore

$$\int \phi_{\mathbf{k}',A}^{\dagger}\phi_{\mathbf{k},A}d^{3}x = \delta_{\mathbf{k},\mathbf{k}'} \tag{12}$$

The same result follows for $\phi_{\mathbf{k},B^{\dagger}}$ by just replacing kx by $-kx$ throughout the derivation, so

$$\int \phi_{\mathbf{k},B^{\dagger}}^{\dagger}\phi_{\mathbf{k},B^{\dagger}}d^{3}x = \delta_{\mathbf{k},\mathbf{k}'} \tag{13}$$

For mixed terms, we have

$$\int \phi_{\mathbf{k}',A}^{\dagger}\phi_{\mathbf{k},B^{\dagger}}d^{3}x = \frac{1}{V}\int e^{i(k'+k)x}d^{3}x \tag{14}$$

In this case, the exponent cannot be zero, so the integral always comes out to zero, so that

$$\int \phi_{\mathbf{k}',A}^{\dagger}\phi_{\mathbf{k},B^{\dagger}}d^{3}x = 0 \tag{15}$$