

KLEIN-GORDON EQUATION: ORTHONORMALITY OF SOLUTIONS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.2.

The plane wave solutions of the Klein-Gordon equation are

$$(1) \quad \phi = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(A_{\mathbf{k}} e^{-ikx} + B_{\mathbf{k}}^{\dagger} e^{ikx} \right)$$

We can redefine a couple of terms by introducing

$$(2) \quad \phi_{\mathbf{k},A} \equiv \frac{e^{-ikx}}{\sqrt{V}}$$

$$(3) \quad \phi_{\mathbf{k},B^{\dagger}} \equiv \frac{e^{ikx}}{\sqrt{V}}$$

Then

$$(4) \quad \phi = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(A_{\mathbf{k}} \phi_{\mathbf{k},A} + B_{\mathbf{k}}^{\dagger} \phi_{\mathbf{k},B^{\dagger}} \right)$$

The $\phi_{\mathbf{k},A}$ and $\phi_{\mathbf{k},B^{\dagger}}$ are orthonormal functions. We have

$$(5) \quad \int \phi_{\mathbf{k},A}^{\dagger} \phi_{\mathbf{k}',A} d^3x = \frac{1}{V} \int e^{i(k'-k)x} d^3x$$

where the integral is over the volume V , and the wavelengths of the plane waves fit an integral number of times within V , so that the amplitudes of the waves at the boundaries are all zero. The four-vector k is defined as

$$(6) \quad k = [\omega_{\mathbf{k}}, \mathbf{k}]$$

If $k' = k$, the integrand is 1 and is integrated over V , so the result is

$$(7) \quad \int \phi_{\mathbf{k},A}^\dagger \phi_{\mathbf{k},A} d^3x = \frac{1}{V} \int e^{i(k'-k)x} d^3x$$

$$(8) \quad = \frac{V}{V} = 1$$

If $k' \neq k$, consider the integral over $x^1 = x$ (for the purposes of this derivation only, x refers to the single x dimension of the 3-vector \mathbf{x} and should not be confused with the four-vector x used in 1):

(9)

$$\int \phi_{\mathbf{k}',A}^\dagger \phi_{\mathbf{k},A} dx = \frac{1}{V} e^{i(\omega_{\mathbf{k}'} - \omega_{\mathbf{k}})t} e^{-i(k_y - k'_y)y} e^{-i(k_z - k'_z)z} \int e^{-i(k_x - k'_x)x} dx$$

$$(10) \quad = -\frac{1}{i(k_x - k'_x)V} e^{i(\omega_{\mathbf{k}'} - \omega_{\mathbf{k}})t} e^{-i(k_y - k'_y)y} e^{-i(k_z - k'_z)z} \left[e^{-i(k_x - k'_x)x} \right]_{x=x_0}^{x=x_1}$$

$$(11) \quad = 0$$

where x_0 and x_1 are the x limits of V , where by assumption the wave amplitude is zero. Therefore

$$(12) \quad \int \phi_{\mathbf{k}',A}^\dagger \phi_{\mathbf{k},A} d^3x = \delta_{\mathbf{k},\mathbf{k}'}$$

The same result follows for $\phi_{\mathbf{k},B^\dagger}$ by just replacing kx by $-kx$ throughout the derivation, so

$$(13) \quad \int \phi_{\mathbf{k},B^\dagger}^\dagger \phi_{\mathbf{k},B^\dagger} d^3x = \delta_{\mathbf{k},\mathbf{k}'}$$

For mixed terms, we have

$$(14) \quad \int \phi_{\mathbf{k}',A}^\dagger \phi_{\mathbf{k},B^\dagger} d^3x = \frac{1}{V} \int e^{i(k'+k)x} d^3x$$

In this case, the exponent cannot be zero, so the integral always comes out to zero, so that

$$(15) \quad \int \phi_{\mathbf{k}',A}^\dagger \phi_{\mathbf{k},B^\dagger} d^3x = 0$$

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