

KLEIN-GORDON EQUATION: PROBABILITY DENSITY AND CURRENT

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.3.

In non-relativistic quantum mechanics governed by the Schrödinger equation, the probability density is given by

$$(1) \quad \rho = \Psi^\dagger \Psi$$

and the probability current is given by (generalizing our earlier result to 3-d and using natural units):

$$(2) \quad \mathbf{J} = \frac{i}{2m} \left(\Psi \nabla \Psi^\dagger - \Psi^\dagger \nabla \Psi \right)$$

The continuity equation for probability is then

$$(3) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

We'll now look at how these results appear in relativistic quantum mechanics, using the Klein-Gordon equation:

$$(4) \quad \frac{\partial^2 \phi}{\partial t^2} = (\nabla^2 - \mu^2) \phi = 0$$

We can multiply this equation by ϕ^\dagger and then subtract the hermitian conjugate of the result from the original equation to get

$$(5) \quad \phi^\dagger \frac{\partial^2 \phi}{\partial t^2} - \phi \frac{\partial^2 \phi^\dagger}{\partial t^2} = \phi^\dagger (\nabla^2 - \mu^2) \phi - \phi (\nabla^2 - \mu^2) \phi^\dagger$$

$$(6) \quad = \phi^\dagger \nabla^2 \phi - \phi \nabla^2 \phi^\dagger$$

The LHS can be written as

$$(7) \quad \phi^\dagger \frac{\partial^2 \phi}{\partial t^2} - \phi \frac{\partial^2 \phi^\dagger}{\partial t^2} = \frac{\partial}{\partial t} \left(\phi^\dagger \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^\dagger}{\partial t} \right)$$

(use the product rule on the RHS and cancel terms).

The RHS of 6 can be written as (use the product rule again):

$$(8) \quad \phi^\dagger \nabla^2 \phi - \phi \nabla^2 \phi^\dagger = \nabla \cdot (\phi^\dagger \nabla \phi - \phi \nabla \phi^\dagger)$$

We can write this as a continuity equation for the Klein-Gordon equation, with the following definitions:

$$(9) \quad \rho \equiv i \left(\phi^\dagger \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^\dagger}{\partial t} \right)$$

$$(10) \quad \mathbf{j} = -i \left(\phi^\dagger \nabla \phi - \phi \nabla \phi^\dagger \right)$$

[The extra i is introduced to make ρ and \mathbf{j} real. Note that the factor within the parentheses in both expressions is a complex quantity minus its complex conjugate, which always gives a pure imaginary term. Thus multiplying by i ensures the result is real.]

Then

$$(11) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

We can put this in 4-vector form if we use (for some 3-vector \mathbf{A}):

$$(12) \quad \nabla \cdot \mathbf{A} = -\partial^i a_i$$

where the implied sum over i is from $i = 1$ to $i = 3$ (spatial coordinates), and the minus sign appears because we've raised the index on ∂^i . If we define

$$(13) \quad j_i = i \left(\phi^\dagger \partial_i \phi - \phi \partial_i \phi^\dagger \right)$$

(that is, the negative of 10), then $\nabla \cdot \mathbf{j} = \partial^i j_i$. To make j_μ into a 4-vector, we add $j_0 = \rho$ and we get

$$(14) \quad \frac{\partial j_0}{\partial t} + \partial^i j_i = \partial^\mu j_\mu = 0$$

[Note that my definition of j_i is the negative of the middle term in Klauber's equation 3-21, although raising the i index agrees with the last term in 3-21. I can't see how his middle and last equations for j_i and j^i can both be right, since raising the i in the middle equation for j_i merely raises the $\phi_{,i}$ to $\phi^{,i}$ without changing the sign.]

The curious thing about the Klein-Gordon equation is that its probability density ρ in 9 need not be positive, depending on the values of ϕ and its time

derivative. To see how this can affect the physical meaning of the equation, consider the general plane wave solution to the Klein-Gordon equation

$$(15) \quad \phi = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(A_{\mathbf{k}} e^{-ikx} + B_{\mathbf{k}}^{\dagger} e^{ikx} \right)$$

Klauber explores this starting with his equation 3-24, where he takes a test solution in which all $B_{\mathbf{k}}^{\dagger} = 0$ and shows that $\int \rho d^3x = \sum_{\mathbf{k}} |A_{\mathbf{k}}|^2 = 1$ so that in this case, the total probability of finding the system in some state is +1 as it should be. Let's see what happens if we take all $A_{\mathbf{k}} = 0$. In that case, 9 becomes

$$(16) \quad \rho = i \left[\sum_{\mathbf{k}} \frac{B_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}} e^{-ikx} \right] \left[\sum_{\mathbf{k}'} \frac{i\omega_{\mathbf{k}'} B_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}} e^{ik'x} \right] -$$

$$i \left[\sum_{\mathbf{k}'} \frac{B_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}} e^{ik'x} \right] \left[\sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}} B_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}} e^{-ikx} \right]$$

$$(17) \quad = - \left[\sum_{\mathbf{k}} \frac{B_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}} e^{-ikx} \right] \left[\sum_{\mathbf{k}'} \frac{\omega_{\mathbf{k}'} B_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}} e^{ik'x} \right] -$$

$$(18) \quad \left[\sum_{\mathbf{k}'} \frac{B_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}} e^{ik'x} \right] \left[\sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}} B_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}} e^{-ikx} \right]$$

We now wish to calculate $\int \rho d^3x$. We can use the orthonormality of solutions to do the integral. We have

$$(19) \quad \frac{1}{V} \int e^{i(k'-k)x} d^3x = \delta_{\mathbf{k},\mathbf{k}'}$$

We get

$$(20) \quad - \int \left[\sum_{\mathbf{k}} \frac{B_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}} e^{-ikx} \right] \left[\sum_{\mathbf{k}'} \frac{\omega_{\mathbf{k}'} B_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}} e^{ik'x} \right] d^3x = - \sum_{\mathbf{k}} \frac{|B_{\mathbf{k}}|^2}{2}$$

$$(21) \quad - \int \left[\sum_{\mathbf{k}'} \frac{B_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}} e^{ik'x} \right] \left[\sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}} B_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}} e^{-ikx} \right] d^3x = - \sum_{\mathbf{k}} \frac{|B_{\mathbf{k}}|^2}{2}$$

$$(22) \quad \int \rho d^3x = - \sum_{\mathbf{k}} |B_{\mathbf{k}}|^2$$

Thus the total probability of finding the system in one of the state \mathbf{k} is negative.

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