

## KLEIN-GORDON EQUATION FOR FIELDS; DERIVATION FROM THE LAGRANGIAN

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the [auxiliary blog](#).

Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.4.

We originally arrived at the Klein-Gordon equation

$$(1) \quad (\square^2 + \mu^2) \phi = 0$$

by converting the relativistic energy equation

$$(2) \quad E^2 = p^2 + m^2$$

to quantum operator form. The function  $\phi$  that is a solution of this equation is a relativistic quantum *state*, that is, it's the relativistic analogue of the state  $\Psi$  that is the solution of the Schrödinger equation.  $\phi$  can represent a relativistic quantum particle in the same way that  $\Psi$  represents a non-relativistic quantum particle.

To get a field theory, that is, a theory where  $\phi(x)$  represents a field value and the four-vector  $x$  is a set of labels of points in space-time (rather than the spatial coordinates  $x^i$  representing the position of a particle), we need to go back to the classical field theory and convert it to a quantum theory. The Euler-Lagrange equations for a classical field are

$$(3) \quad \frac{\partial \mathcal{L}}{\partial \phi^r} - \frac{\partial}{\partial q^\mu} \left( \frac{\partial \mathcal{L}}{\partial \phi_{, \mu}^r} \right) = 0$$

where  $\phi^r$  is the  $r$ th field,  $q^\mu$  is the  $\mu$ th coordinate in spacetime and  $\mathcal{L}$  is the Lagrangian (strictly, the Lagrangian density). At this point, most sources I've seen magically produce  $\mathcal{L}$  seemingly out of thin air, although some books (such as Klauber) mention that the Lagrangian does come out of classical field theory. I may delve into this at some point, but in order not to slow things down too much, I'll just quote the Lagrangian here:

$$(4) \quad \mathcal{L}_0 = K (\partial_\alpha \phi \partial^\alpha \phi - \mu^2 \phi^2)$$

The superscript 0 on  $\mathcal{L}_0^0$  indicates that we're dealing with a *scalar* field (as opposed to a field with spin) and the subscript 0 indicates that it's a *free* field (no potential terms).  $K$  is a constant. The Legendre transformation gives the Hamiltonian density for the field as

$$(5) \quad \mathcal{H}_0^0 = \pi_0^0 \dot{\phi} - \mathcal{L}_0^0$$

where the conjugate momentum is, from 4

$$(6) \quad \pi_0^0 = \frac{\partial \mathcal{L}_0^0}{\partial (\partial_0 \phi)} = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}} = 2K\dot{\phi}$$

Therefore

$$(7) \quad \mathcal{H}_0^0 = 2K\dot{\phi}\dot{\phi} - K(\partial_\alpha \partial^\alpha \phi - \mu^2 \phi^2)$$

$$(8) \quad = K(\dot{\phi}\dot{\phi} + \nabla\phi \cdot \nabla\phi + \mu^2 \phi^2)$$

At this point, there's a bit of (informed) hand-waving, where we realize that since the wave function  $\Psi$  in Schrödinger quantum mechanics is a complex function, it's most likely that  $\phi$  in field theory is also complex. However, if we wish to retain the meaning of the Hamiltonian as the energy density, it must be real, so we need to combine  $\phi$  and  $\phi^\dagger$  in such a way as to give a real Hamiltonian and Lagrangian. We can do this by noticing that  $\phi$  and its derivatives always appear in squared terms in 4 and 8, which suggests replacing these terms by products of a term with its complex conjugate. So we try (taking  $K = 1$ )

$$(9) \quad \mathcal{L}_0^0 = \partial_\alpha \phi^\dagger \partial^\alpha \phi - \mu^2 \phi^\dagger \phi$$

$$(10) \quad = \dot{\phi}^\dagger \dot{\phi} - \nabla\phi^\dagger \cdot \nabla\phi - \mu^2 \phi^\dagger \phi$$

Calculating the Hamiltonian is a bit trickier, for we need to realize that with a complex field, we actually have *two* fields, since the real and imaginary parts are two separate functions. In fact, the way this is handled is to treat  $\phi$  and  $\phi^\dagger$  as the separate fields. Then our original notation for the Hamiltonian is

$$(11) \quad \mathcal{H}_0^0 = \pi_r \dot{\phi}^r - \mathcal{L}_0^0$$

where there is a sum over  $r$ , the index specifying which field we're talking about. Here, this becomes

$$(12) \quad \mathcal{H}_0^0 = \pi_0^0 \dot{\phi} + \pi_0^{0\dagger} \dot{\phi}^\dagger - \mathcal{L}_0^0$$

From 6 and 10 we have

$$(13) \quad \pi_0^0 = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}} = \dot{\phi}^\dagger$$

$$(14) \quad \pi_0^{0\dagger} = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}^\dagger} = \dot{\phi}$$

$$(15) \quad \mathcal{H}_0^0 = \dot{\phi}^\dagger \dot{\phi} + \dot{\phi} \dot{\phi}^\dagger - \left[ \dot{\phi}^\dagger \phi - \nabla \phi^\dagger \cdot \nabla \phi - \mu^2 \phi^\dagger \phi \right]$$

$$(16) \quad = \dot{\phi} \dot{\phi}^\dagger + \nabla \phi^\dagger \cdot \nabla \phi + \mu^2 \phi^\dagger \phi$$

Using 10 in 3 we get, for  $\phi^r = \phi$ :

$$(17) \quad \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial q^\mu} \left( \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) = -\mu^2 \phi^\dagger - \partial_0 (\partial^0 \phi^\dagger) + \partial_i (\partial^i \phi^\dagger) = 0$$

$$(18) \quad (\partial_\mu \partial^\mu + \mu^2) \phi^\dagger = (\square^2 + \mu^2) \phi^\dagger = 0$$

Taking  $\phi^r = \phi^\dagger$  gives the conjugate equation

$$(19) \quad (\partial_\mu \partial^\mu + \mu^2) \phi = (\square^2 + \mu^2) \phi = 0$$

This is the Klein-Gordon equation, but the crucial difference is that the function  $\phi$  here is a *field* rather than the function  $\phi$  in our earlier solution, which represented a state of a particle or superposition of particles. In the former solution,  $\phi$  was a direct analogue of the non-relativistic wave function  $\Psi$ ; in our present case, it represents the value of the quantum field at each point  $q^\mu$  in spacetime.

However, the two solutions are *mathematically* equivalent, so they must have the same solution, which we can write as

$$(20) \quad \phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a(\mathbf{k}) e^{-ikx} + \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} b^\dagger(\mathbf{k}) e^{ikx}$$

$$(21) \quad \equiv \phi^+ + \phi^-$$

$$(22) \quad \phi^\dagger(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a^\dagger(\mathbf{k}) e^{ikx} + \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} b(\mathbf{k}) e^{-ikx}$$

$$(23) \quad \equiv \phi^{\dagger+} + \phi^{\dagger-}$$

[Take care to distinguish between the  $+$  and  $\dagger$  symbols in the superscripts.] We've written the coefficient  $a$  and  $b$  using lowercase rather than the uppercase we used for the particle theory, as they turn out to be operators rather than simply numerical constants.

We can work out the constant  $\mu$  in 19 in traditional units by applying this equation to one component of the solution. Since 20 is a sum of terms, each of which is a solution, we can look at just one term, say  $\phi_{\mathbf{k}} = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a(\mathbf{k}) e^{-ikx}$ . Then

$$(24) \quad (\partial_{\mu}\partial^{\mu} + \mu^2) \phi_{\mathbf{k}} = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a(\mathbf{k}) (\partial_{\mu}\partial^{\mu} + \mu^2) e^{-ikx}$$

$$(25) \quad = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a(\mathbf{k}) \left( (-i)^2 k_{\mu}k^{\mu} + \mu^2 \right) e^{-ikx}$$

$$(26) \quad = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a(\mathbf{k}) \left( -k_{\mu}k^{\mu} + \mu^2 \right) e^{-ikx}$$

So we must have

$$(27) \quad \mu^2 = k_{\mu}k^{\mu}$$

In cgs units, the 4-vector for spacetime is

$$(28) \quad x^{\mu} = [ct, \mathbf{x}]$$

The four-vector  $k^{\mu}$  must be

$$(29) \quad k^{\mu} = \left[ \frac{E_{\mathbf{k}}}{c\hbar}, \frac{\mathbf{p}_{\mathbf{k}}}{\hbar} \right]$$

[We get this by multiplying by factors of  $c$  and  $\hbar$  to make the product  $k_{\mu}x^{\mu}$  dimensionless, as it must be since it is an exponent. Remember that the dimensions of  $\hbar$  in cgs are (energy)  $\times$  (time).]

So

$$(30) \quad k_{\mu}k^{\mu} = \frac{1}{\hbar^2} \left( \frac{E_{\mathbf{k}}^2}{c^2} - p_{\mathbf{k}}^2 \right)$$

Using the relativistic relation  $E^2 = p^2c^2 + m^2c^4$  we get

$$(31) \quad k_{\mu}k^{\mu} = \frac{1}{\hbar^2} (p_{\mathbf{k}}^2 + m^2c^2 - p_{\mathbf{k}}^2)$$

$$(32) \quad = m^2 \frac{c^2}{\hbar^2} = \mu^2$$

PINGBACKS

- Pingback: Klein-Gordon equation: continuous solutions
- Pingback: Klein-Gordon equation from the Heisenberg picture
- Pingback: Klein-Gordon equation: commutators
- Pingback: Free scalar Hamiltonian as an integral of the Hamiltonian density
- Pingback: Probability density in a Klein-Gordon field
- Pingback: Momentum of a free scalar Klein-Gordon field
- Pingback: Real Klein-Gordon fields
- Pingback: Klein-Gordon equation with anticommutators
- Pingback: Dirac equation in quantum field theory: Lagrangian density
- Pingback: Noether's theorem
- Pingback: Stress-energy tensor from Noether's theorem
- Pingback: Klein-Gordon solutions from harmonic oscillator
- Pingback: Klein-Gordon equation from harmonic oscillator: Hamiltonian, creation and
- Pingback: Green's function for Klein-Gordon equation
- Pingback: Klein-Gordon Green's function as a Fourier transform