

## KLEIN-GORDON EQUATION FOR FIELDS; DERIVATION FROM THE LAGRANGIAN

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.4.

We originally arrived at the Klein-Gordon equation

$$(\square^2 + \mu^2) \phi = 0 \tag{1}$$

by converting the relativistic energy equation

$$E^2 = p^2 + m^2 \tag{2}$$

to quantum operator form. The function  $\phi$  that is a solution of this equation is a relativistic quantum *state*, that is, it's the relativistic analogue of the state  $\Psi$  that is the solution of the Schrödinger equation.  $\phi$  can represent a relativistic quantum particle in the same way that  $\Psi$  represents a non-relativistic quantum particle.

To get a field theory, that is, a theory where  $\phi(x)$  represents a field value and the four-vector  $x$  is a set of labels of points in space-time (rather than the spatial coordinates  $x^i$  representing the position of a particle), we need to go back to the classical field theory and convert it to a quantum theory. The Euler-Lagrange equations for a classical field are

$$\frac{\partial \mathcal{L}}{\partial \phi^r} - \frac{\partial}{\partial q^\mu} \left( \frac{\partial \mathcal{L}}{\partial \phi^r_{,\mu}} \right) = 0 \tag{3}$$

where  $\phi^r$  is the  $r$ th field,  $q^\mu$  is the  $\mu$ th coordinate in spacetime and  $\mathcal{L}$  is the Lagrangian (strictly, the Lagrangian density). At this point, most sources I've seen magically produce  $\mathcal{L}$  seemingly out of thin air, although some books (such as Klauber) mention that the Lagrangian does come out of classical field theory. I may delve into this at some point, but in order not to slow things down too much, I'll just quote the Lagrangian here:

$$\mathcal{L}_0^0 = K (\partial_\alpha \phi \partial^\alpha \phi - \mu^2 \phi^2) \tag{4}$$

The superscript 0 on  $\mathcal{L}_0^0$  indicates that we're dealing with a *scalar* field (as opposed to a field with spin) and the subscript 0 indicates that it's a *free*

field (no potential terms).  $K$  is a constant. The Legendre transformation gives the Hamiltonian density for the field as

$$\mathcal{H}_0^0 = \pi_0^0 \dot{\phi} - \mathcal{L}_0^0 \quad (5)$$

where the conjugate momentum is, from 4

$$\pi_0^0 = \frac{\partial \mathcal{L}_0^0}{\partial (\partial_0 \phi)} = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}} = 2K\dot{\phi} \quad (6)$$

Therefore

$$\mathcal{H}_0^0 = 2K\dot{\phi}\dot{\phi} - K(\partial_\alpha \partial^\alpha \phi - \mu^2 \phi^2) \quad (7)$$

$$= K(\dot{\phi}\dot{\phi} + \nabla\phi \cdot \nabla\phi + \mu^2 \phi^2) \quad (8)$$

At this point, there's a bit of (informed) hand-waving, where we realize that since the wave function  $\Psi$  in Schrödinger quantum mechanics is a complex function, it's most likely that  $\phi$  in field theory is also complex. However, if we wish to retain the meaning of the Hamiltonian as the energy density, it must be real, so we need to combine  $\phi$  and  $\phi^\dagger$  in such a way as to give a real Hamiltonian and Lagrangian. We can do this by noticing that  $\phi$  and its derivatives always appear in squared terms in 4 and 8, which suggests replacing these terms by products of a term with its complex conjugate. So we try (taking  $K = 1$ )

$$\mathcal{L}_0^0 = \partial_\alpha \phi^\dagger \partial^\alpha \phi - \mu^2 \phi^\dagger \phi \quad (9)$$

$$= \dot{\phi}^\dagger \dot{\phi} - \nabla\phi^\dagger \cdot \nabla\phi - \mu^2 \phi^\dagger \phi \quad (10)$$

Calculating the Hamiltonian is a bit trickier, for we need to realize that with a complex field, we actually have *two* fields, since the real and imaginary parts are two separate functions. In fact, the way this is handled is to treat  $\phi$  and  $\phi^\dagger$  as the separate fields. Then our original notation for the Hamiltonian is

$$\mathcal{H}_0^0 = \pi_r \dot{\phi}^r - \mathcal{L}_0^0 \quad (11)$$

where there is a sum over  $r$ , the index specifying which field we're talking about. Here, this becomes

$$\mathcal{H}_0^0 = \pi_0^0 \dot{\phi} + \pi_0^{0\dagger} \dot{\phi}^\dagger - \mathcal{L}_0^0 \quad (12)$$

From 6 and 10 we have

$$\pi_0^0 = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}} = \dot{\phi}^\dagger \quad (13)$$

$$\pi_0^{0\dagger} = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}^\dagger} = \dot{\phi} \quad (14)$$

$$\mathcal{H}_0^0 = \dot{\phi}^\dagger \dot{\phi} + \dot{\phi} \dot{\phi}^\dagger - \left[ \dot{\phi}^\dagger \phi - \nabla \phi^\dagger \cdot \nabla \phi - \mu^2 \phi^\dagger \phi \right] \quad (15)$$

$$= \dot{\phi} \dot{\phi}^\dagger + \nabla \phi^\dagger \cdot \nabla \phi + \mu^2 \phi^\dagger \phi \quad (16)$$

Using 10 in 3 we get, for  $\phi^r = \phi$ :

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial q^\mu} \left( \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) = -\mu^2 \phi^\dagger - \partial_0 (\partial^0 \phi^\dagger) + \partial_i (\partial^i \phi^\dagger) = 0 \quad (17)$$

$$(\partial_\mu \partial^\mu + \mu^2) \phi^\dagger = (\square^2 + \mu^2) \phi^\dagger = 0 \quad (18)$$

Taking  $\phi^r = \phi^\dagger$  gives the conjugate equation

$$(\partial_\mu \partial^\mu + \mu^2) \phi = (\square^2 + \mu^2) \phi = 0 \quad (19)$$

This is the Klein-Gordon equation, but the crucial difference is that the function  $\phi$  here is a *field* rather than the function  $\phi$  in our earlier solution, which represented a state of a particle or superposition of particles. In the former solution,  $\phi$  was a direct analogue of the non-relativistic wave function  $\Psi$ ; in our present case, it represents the value of the quantum field at each point  $q^\mu$  in spacetime.

However, the two solutions are *mathematically* equivalent, so they must have the same solution, which we can write as

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V \omega_{\mathbf{k}}}} a(\mathbf{k}) e^{-ikx} + \sum_{\mathbf{k}} \frac{1}{\sqrt{2V \omega_{\mathbf{k}}}} b^\dagger(\mathbf{k}) e^{ikx} \quad (20)$$

$$\equiv \phi^+ + \phi^- \quad (21)$$

$$\phi^\dagger(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V \omega_{\mathbf{k}}}} a^\dagger(\mathbf{k}) e^{ikx} + \sum_{\mathbf{k}} \frac{1}{\sqrt{2V \omega_{\mathbf{k}}}} b(\mathbf{k}) e^{-ikx} \quad (22)$$

$$\equiv \phi^{\dagger+} + \phi^{\dagger-} \quad (23)$$

[Take care to distinguish between the  $+$  and  $\dagger$  symbols in the superscripts.] We've written the coefficient  $a$  and  $b$  using lowercase rather than the uppercase we used for the particle theory, as they turn out to be operators rather than simply numerical constants.

We can work out the constant  $\mu$  in 19 in traditional units by applying this equation to one component of the solution. Since 20 is a sum

of terms, each of which is a solution, we can look at just one term, say  $\phi_{\mathbf{k}} = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a(\mathbf{k}) e^{-ikx}$ . Then

$$(\partial_{\mu}\partial^{\mu} + \mu^2)\phi_{\mathbf{k}} = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a(\mathbf{k}) (\partial_{\mu}\partial^{\mu} + \mu^2) e^{-ikx} \quad (24)$$

$$= \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a(\mathbf{k}) \left( (-i)^2 k_{\mu}k^{\mu} + \mu^2 \right) e^{-ikx} \quad (25)$$

$$= \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a(\mathbf{k}) \left( -k_{\mu}k^{\mu} + \mu^2 \right) e^{-ikx} \quad (26)$$

So we must have

$$\mu^2 = k_{\mu}k^{\mu} \quad (27)$$

In cgs units, the 4-vector for spacetime is

$$x^{\mu} = [ct, \mathbf{x}] \quad (28)$$

The four-vector  $k^{\mu}$  must be

$$k^{\mu} = \left[ \frac{E_{\mathbf{k}}}{c\hbar}, \frac{\mathbf{p}_{\mathbf{k}}}{\hbar} \right] \quad (29)$$

[We get this by multiplying by factors of  $c$  and  $\hbar$  to make the product  $k_{\mu}x^{\mu}$  dimensionless, as it must be since it is an exponent. Remember that the dimensions of  $\hbar$  in cgs are (energy)  $\times$  (time).]

So

$$k_{\mu}k^{\mu} = \frac{1}{\hbar^2} \left( \frac{E_{\mathbf{k}}^2}{c^2} - p_{\mathbf{k}}^2 \right) \quad (30)$$

Using the relativistic relation  $E^2 = p^2c^2 + m^2c^4$  we get

$$k_{\mu}k^{\mu} = \frac{1}{\hbar^2} (p_{\mathbf{k}}^2 + m^2c^2 - p_{\mathbf{k}}^2) \quad (31)$$

$$= m^2 \frac{c^2}{\hbar^2} = \mu^2 \quad (32)$$

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