

## KLEIN-GORDON EQUATION: CONTINUOUS SOLUTIONS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.5.

The Klein-Gordon equation

$$(\square^2 + \mu^2) \phi = 0 \tag{1}$$

has discrete plane-wave solutions of form

$$\phi = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left( A_{\mathbf{k}} e^{-ikx} + B_{\mathbf{k}}^{\dagger} e^{ikx} \right) \tag{2}$$

where  $V$  is the volume containing the waves. The frequencies  $\omega_{\mathbf{k}}$  and wave numbers  $\mathbf{k}$  are constrained by the requirement that the waves must fit within  $V$ , so that their amplitudes are zero at the boundary.

A more general solution which allows all frequencies of waves and is not confined to a particular volume is given by

$$\phi(x) = \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} a(\mathbf{k}) e^{-ikx} + \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} b^{\dagger}(\mathbf{k}) e^{ikx} \tag{3}$$

$$\equiv \phi^{+} + \phi^{-} \tag{4}$$

$$\phi^{\dagger}(x) = \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} b(\mathbf{k}) e^{-ikx} + \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} a^{\dagger}(\mathbf{k}) e^{ikx} \tag{5}$$

$$\equiv \phi^{\dagger+} + \phi^{\dagger-} \tag{6}$$

To verify that these are solutions of the Klein-Gordon equation 1, we note that the operator  $\square^2$  acts only on  $x$  which appears only in the exponential factors inside the integrals. Thus we get

$$\begin{aligned}
(\square^2 + \mu^2) \phi &= \int \frac{d^3k}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}} a(\mathbf{k}) \left( (-i)^2 k_\mu k^\mu + \mu^2 \right) e^{-ikx} + \\
&\int \frac{d^3k}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}} b^\dagger(\mathbf{k}) \left( i^2 k_\mu k^\mu + \mu^2 \right) e^{ikx} \quad (7)
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{d^3k}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}} a(\mathbf{k}) (\mu^2 - k_\mu k^\mu) e^{-ikx} + \\
&\int \frac{d^3k}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}} b^\dagger(\mathbf{k}) (\mu^2 - k_\mu k^\mu) e^{ikx} \quad (8)
\end{aligned}$$

As with the discrete case, we can use the relativistic condition that  $k_\mu k^\mu = m^2 \frac{c^2}{\hbar^2}$  to find that  $\mu = \frac{mc}{\hbar}$  (or just  $m$  in natural units), so the integral solution does indeed satisfy the Klein-Gordon equation.

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