KLEIN-GORDON EQUATION FROM THE HEISENBERG PICTURE

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.6.

The Klein-Gordon equation

$$\left(\Box^2+\mu^2\right)\phi=0 \tag{1}$$

can be derived from the equation of motion of an operator in the Heisenberg picture of quantum mechanics. In the Heisenberg picture, the time dependence has been shifted from the states (where it resides in the Schrödinger picture) to the operators, so that the equation of motion of an operator Q is given by

$$i\frac{dQ}{dt}=[Q,H] \tag{2}$$

[Here, we're assuming that the Schrödinger operator Q^S has no explicit time dependence, so that the $\frac{\partial Q}{\partial t}$ term is zero.]

If we regard the field ϕ as an operator, then we get

$$i\frac{\partial\phi}{\partial t}=[\phi,H] \tag{3}$$

To calculate this, we need to know the full Hamiltonian H, which is the integral of the Hamiltonian density $\mathscr{H}$ over space. We worked out the Lagrangian density $\mathscr{L}$ and $\mathscr{H}$ earlier:

$$\mathscr{L} = \dot{\phi}^{\dagger}\dot{\phi}-\nabla\phi^{\dagger}\cdot\nabla\phi-\mu^2\phi^{\dagger}\phi \tag{4}$$

$$\mathscr{H} = \dot{\phi}^{\dagger}\dot{\phi}+\nabla\phi^{\dagger}\cdot\nabla\phi+\mu^2\phi^{\dagger}\phi \tag{5}$$

Using the definition of conjugate momentum

$$\pi=\frac{\partial\mathscr{L}}{\partial\dot{\phi}}=\dot{\phi}^{\dagger} \tag{6}$$

we can write this as

$$\mathscr{H} = \pi^\dagger \pi + \nabla \phi^\dagger \cdot \nabla \phi + \mu^2 \phi^\dagger \phi \tag{7}$$

We therefore need to calculate

$$[\phi, H] = \left[\phi, \int d^3\mathbf{x}' \left(\pi^\dagger \pi + \nabla' \phi^\dagger \cdot \nabla' \phi + \mu^2 \phi^\dagger \phi\right)\right] \tag{8}$$

where everything in the integrand is a function of the spatial integration variable $\mathbf{x}'$, and the ϕ outside the integral is a function of some fixed location $\mathbf{x}$.

From the commutation relations

$$[\phi^r, \pi_s] = i\delta^r_s \delta^3 \left(\mathbf{x} - \mathbf{x}'\right) \tag{9}$$

$$[\phi^r, \phi^s] = 0 \tag{10}$$

$$[\pi_r, \pi_s] = 0 \tag{11}$$

we know that the ϕ outside the integral commutes with the terms involving ϕ inside the integral, so the only non-zero contribution to the commutator comes from the $\pi^\dagger \pi$ term. That is

$$[\phi, H] = \left[\phi, \int d^3\mathbf{x}' \pi^\dagger \pi\right] \tag{12}$$

From now on, we can regard ϕ as a function of $\mathbf{x}$ (that is, the spatial coordinate that is *not* integrated over) and t (which we won't write to avoid cluttering things) and $\pi^\dagger$ and π as functions of $\mathbf{x}'$ (the integration variable) and t. From the point of view of the integration, then, $\phi = \phi(\mathbf{x})$ is a constant and can be taken inside the integral, so we get

$$\left[\phi, \int d^3\mathbf{x}' \pi^\dagger \pi\right] = \int d^3\mathbf{x}' \left[\phi, \pi^\dagger \pi\right] \tag{13}$$

Using 9, we get (remember that $\left[\phi, \pi^\dagger\right] = 0$ since the field and its complex conjugate are *different* fields, so $r \neq s$ in 9):

$$\left[\phi, \pi^\dagger \pi\right] = \phi \pi^\dagger \pi - \pi^\dagger \pi \phi \tag{14}$$

$$= \pi^\dagger \phi \pi - \pi^\dagger \phi \pi + i\pi^\dagger \delta^3 \left(\mathbf{x} - \mathbf{x}'\right) \tag{15}$$

$$= i\pi^\dagger \left(\mathbf{x}', t\right) \delta^3 \left(\mathbf{x} - \mathbf{x}'\right) \tag{16}$$

Doing the integral gives us

$$i\frac{\partial\phi}{\partial t}=\int d^{3}\mathbf{x}'\left[\phi,\pi^{\dagger}\pi\right]=i\pi^{\dagger}\left(\mathbf{x},t\right) \tag{17}$$

That's the easy bit. We now need to work out the same thing for the conjugate momentum π (actually, we'll deal with $\pi^{\dagger}$). We have

$$i\frac{\partial\pi^{\dagger}}{\partial t} = \left[\pi^{\dagger},H\right] \tag{18}$$

$$= \left[\pi^{\dagger},\int d^{3}\mathbf{x}'\left(\pi^{\dagger}\pi+\nabla'\phi^{\dagger}\cdot\nabla'\phi+\mu^{2}\phi^{\dagger}\phi\right)\right] \tag{19}$$

The commutator with the $\pi^{\dagger}\pi$ term in the integrand gives zero, so we're left with

$$i\frac{\partial\pi^{\dagger}}{\partial t}=\left[\pi^{\dagger},\int d^{3}\mathbf{x}'\left(\nabla'\phi^{\dagger}\cdot\nabla'\phi+\mu^{2}\phi^{\dagger}\phi\right)\right] \tag{20}$$

This time, $\pi^{\dagger}$ is a function of the non-integrated position $\mathbf{x}$ and the ϕ terms inside the integral are functions of $\mathbf{x}'$, so we can take the commutator inside the integral as before

$$i\frac{\partial\pi^{\dagger}}{\partial t}=\int d^{3}\mathbf{x}'\left[\pi^{\dagger},\left(\nabla'\phi^{\dagger}\cdot\nabla'\phi+\mu^{2}\phi^{\dagger}\phi\right)\right] \tag{21}$$

We'll look at the the first term, involving the gradients. From the point of view of the ∇' operator, $\pi^{\dagger}$ is a constant (since it depends on $\mathbf{x}$, not $\mathbf{x}'$) so we can move it in and out of the derivatives with no effect. Therefore

$$\left[\pi^{\dagger},\nabla'\phi^{\dagger}\cdot\nabla'\phi\right]=\nabla'\left(\pi^{\dagger}\phi^{\dagger}\right)\cdot\nabla'\phi-\nabla'\phi^{\dagger}\cdot\nabla'\left(\pi^{\dagger}\phi\right) \tag{22}$$

$$=\nabla'\left(\pi^{\dagger}\phi^{\dagger}\right)\cdot\nabla'\phi-\nabla'\phi^{\dagger}\cdot\nabla'\left(\phi\pi^{\dagger}\right) \tag{23}$$

$$=\nabla'\left(\pi^{\dagger}\phi^{\dagger}\right)\cdot\nabla'\phi-\nabla'\left(\phi^{\dagger}\pi^{\dagger}\right)\cdot\nabla'\phi \tag{24}$$

where we've juggled the last term by using the fact that ϕ commutes with both $\phi^{\dagger}$ and $\pi^{\dagger}$. Now we have from 9

$$\nabla'\left(\phi^{\dagger}\pi^{\dagger}\right)\cdot\nabla'\phi=\nabla'\left(\pi^{\dagger}\phi^{\dagger}\right)\cdot\nabla'\phi+i\nabla'\left(\delta^{3}\left(\mathbf{x}-\mathbf{x}'\right)\right)\cdot\nabla'\phi \tag{25}$$

So

$$\nabla'\left(\pi^{\dagger}\phi^{\dagger}\right)\cdot\nabla'\phi-\nabla'\left(\phi^{\dagger}\pi^{\dagger}\right)\cdot\nabla'\phi=-i\nabla'\left(\delta^{3}\left(\mathbf{x}-\mathbf{x}'\right)\right)\cdot\nabla'\phi \tag{26}$$

The last term can be converted using the product rule for two functions f and g:

$$\nabla' \cdot \left(f\nabla' g\right) = \left(\nabla' f\right) \cdot \left(\nabla' g\right) + f\nabla'^2 g \tag{27}$$

In our case,

$$-i\nabla'\left(\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\right)\cdot\nabla'\phi = -i\nabla'\cdot\left(\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\nabla'\phi\right) + i\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\nabla'^2\phi \tag{28}$$

The second term is a divergence which, when integrated, gets converted to a surface integral using Gauss's theorem and since our observation point $\mathbf{x}$ is at a finite location, this integral goes to zero if we let the surface become arbitrarily large. Therefore we're left with only the second term, and we have

$$\int d^3\mathbf{x}'\left[\pi^\dagger, \left(\nabla'\phi^\dagger\cdot\nabla'\phi\right)\right] = i\int d^3\mathbf{x}'\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\nabla'^2\phi \tag{29}$$

$$= i\nabla^2\phi\left(\mathbf{x},t\right) \tag{30}$$

The last term in 21 is a bit easier to deal with. We have

$$\left[\pi^\dagger, \mu^2\phi^\dagger\phi\right] = \mu^2\left(\pi^\dagger\phi^\dagger\phi - \phi^\dagger\phi\pi^\dagger\right) \tag{31}$$

$$= \mu^2\left(\pi^\dagger\phi^\dagger\phi - \phi^\dagger\pi^\dagger\phi\right) \tag{32}$$

$$= \mu^2\left(\pi^\dagger\phi^\dagger\phi - \pi^\dagger\phi^\dagger\phi - i\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\right) \tag{33}$$

$$= -i\mu^2\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{34}$$

Doing the integral, we get

$$\int d^3\mathbf{x}'\left[\pi^\dagger, \mu^2\phi^\dagger\phi\right] = -i\mu^2 \tag{35}$$

Therefore from 30 and 35 we get

$$\frac{\partial\pi^\dagger}{\partial t} = \nabla^2\phi\left(\mathbf{x},t\right) - \mu^2\phi \tag{36}$$

Taking the derivative of 17 with respect to t we get

$$\frac{\partial^2\phi}{\partial t^2} = \frac{\partial\pi^\dagger}{\partial t} \tag{37}$$

$$= \nabla^2\phi\left(\mathbf{x},t\right) - \mu^2\phi \tag{38}$$

$$\left(\Box^2 + \mu^2\right)\phi = 0 \tag{39}$$

Thus we reclaim the Klein-Gordon equation by using the Heisenberg picture and the Hamiltonian and Lagrangian densities we had from before.