

KLEIN-GORDON EQUATION: COMMUTATORS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.7.

The solutions to the Klein-Gordon equation

$$(1) \quad (\square^2 + \mu^2) \phi = 0$$

has discrete plane-wave solutions of form

$$(2) \quad \phi = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(a(\mathbf{k}) e^{-ikx} + b^\dagger(\mathbf{k}) e^{ikx} \right)$$

and continuous plane-wave solutions of form

$$(3) \quad \phi(x) = \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} a(\mathbf{k}) e^{-ikx} + \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} b^\dagger(\mathbf{k}) e^{ikx}$$

The second quantization postulate is that the Poisson brackets from classical field theory translate into commutators in quantum field theory, so that

$$(4) \quad [\phi^r(\mathbf{x}, t), \pi_s(\mathbf{x}', t)] = i\hbar \delta_s^r \delta^3(\mathbf{x} - \mathbf{x}')$$

$$(5) \quad [\phi^r, \phi^s] = 0$$

$$(6) \quad [\pi_r, \pi_s] = 0$$

where the conjugate momentum is, for the free-field scalar Lagrangian density we're using here:

$$(7) \quad \pi_r = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^r} = \dot{\phi}^{r\dagger}$$

In Klauber's section 3.3, he goes through a detailed calculation for the discrete solution 2 to show that applying the commutation relation 4 to this solution results in commutation relations for the a and b operators:

$$(8) \quad [a(\mathbf{k}), a^\dagger(\mathbf{k}')] = [b(\mathbf{k}), b^\dagger(\mathbf{k}')] = \delta_{\mathbf{k}\mathbf{k}'}$$

To abbreviate the notation a bit, we'll write this as

$$(9) \quad [a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = [b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}$$

The derivation of the corresponding commutators for the continuous solution 3 is very similar, although a bit lengthy, but it's useful to review the process.

In the derivation, we have two fields: ϕ and ϕ^\dagger and their conjugate momenta π and π^\dagger . From 3 and 7, and the fact that k is a 4-vector:

$$(10) \quad k = (\omega_{\mathbf{k}}, \mathbf{k})$$

we have

$$(11) \quad \phi(\mathbf{x}, t) = \int \frac{d^3k}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}} a_{\mathbf{k}} e^{-i(\omega_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} + \int \frac{d^3k}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}} b_{\mathbf{k}}^\dagger e^{i(\omega_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})}$$

$$(12) \quad \pi(\mathbf{x}', t) = \int \frac{i\omega_{\mathbf{k}'} d^3k'}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}'}}} a_{\mathbf{k}'}^\dagger e^{i(\omega_{\mathbf{k}'}t - \mathbf{k}' \cdot \mathbf{x}')} + \int \frac{-i\omega_{\mathbf{k}'} d^3k'}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}'}}} b_{\mathbf{k}'} e^{-i(\omega_{\mathbf{k}'}t - \mathbf{k}' \cdot \mathbf{x}')}$$

I've used different integration variables k and k' for the two equations, since we'll be multiplying them and we need to keep the integration variables separate. If we substitute these two equations into the LHS of 4 and multiply them out, we get a long expression which we can write as

$$(13) \quad [\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = \frac{1}{2(2\pi)^3} \int d^3k \int d^3k' \frac{i\omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} X$$

where X is

$$(14) \quad X = [a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] e^{i(\omega_{\mathbf{k}'} - \omega_{\mathbf{k}})t} e^{i(\mathbf{k} \cdot \mathbf{x} - \mathbf{k}' \cdot \mathbf{x}')} -$$

$$(15) \quad [a_{\mathbf{k}}, b_{\mathbf{k}'}] e^{-i(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}})t} e^{i(\mathbf{k} \cdot \mathbf{x} + \mathbf{k}' \cdot \mathbf{x}')} -$$

$$(16) \quad [a_{\mathbf{k}'}^\dagger, b_{\mathbf{k}}^\dagger] e^{i(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}})t} e^{-i(\mathbf{k} \cdot \mathbf{x} + \mathbf{k}' \cdot \mathbf{x}')} +$$

$$(17) \quad [b_{\mathbf{k}'}^\dagger, b_{\mathbf{k}}^\dagger] e^{-i(\omega_{\mathbf{k}'} - \omega_{\mathbf{k}})t} e^{-i(\mathbf{k} \cdot \mathbf{x} - \mathbf{k}' \cdot \mathbf{x}')} -$$

The delta function on the RHS of 4 can be written as an integral:

$$(18) \quad \delta^3(\mathbf{x} - \mathbf{x}') = \frac{1}{(2\pi)^3} \int d^3k e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}$$

Therefore, we must have

$$(19) \quad \frac{1}{2} \int d^3k' \frac{i\omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} X = e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}$$

The RHS of 19 has no time dependence, so the terms in 15 and 16 must be zero, since they contain a factor $e^{\pm i(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}})t}$ which is time-dependent, no matter what \mathbf{k} and \mathbf{k}' are (remember that $\omega_{\mathbf{k}}$ is always positive as it represents a frequency). Therefore

$$(20) \quad [a_{\mathbf{k}}, b_{\mathbf{k}'}] = [a_{\mathbf{k}}^\dagger, b_{\mathbf{k}'}^\dagger] = 0$$

The remaining two terms 14 and 17 are independent of time if $\omega_{\mathbf{k}} = \omega_{\mathbf{k}'}$ which implies that $|\mathbf{k}| = |\mathbf{k}'|$. Choosing

$$(21) \quad [a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = [b_{\mathbf{k}'}, b_{\mathbf{k}}^\dagger] = \delta^3(\mathbf{k} - \mathbf{k}')$$

certainly satisfies this condition as well as the original commutator 4, since

$$(22) \quad \frac{1}{2(2\pi)^3} \int d^3k \int d^3k' \frac{i\omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} X = \frac{i}{2(2\pi)^3} \int d^3k \int d^3k' \delta^3(\mathbf{k} - \mathbf{k}') \left(e^{i(\mathbf{k}\cdot\mathbf{x} - \mathbf{k}'\cdot\mathbf{x}')} + e^{-i(\mathbf{k}\cdot\mathbf{x} - \mathbf{k}'\cdot\mathbf{x}')} \right)$$

$$(23) \quad = \frac{i}{2(2\pi)^3} \int d^3k \left(e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} + e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \right)$$

$$(24) \quad = i\delta^3(\mathbf{x} - \mathbf{x}')$$

It would seem to me that the justification for this choice (rather than a more general condition which gives non-zero commutators for any pair \mathbf{k} and \mathbf{k}' where $|\mathbf{k}| = |\mathbf{k}'|$, and not the more restrictive condition that $\mathbf{k} = \mathbf{k}'$) is that these more general conditions don't satisfy 4.

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