KLEIN-GORDON EQUATION: COMMUTATORS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.7.

The solutions to the Klein-Gordon equation

$$\left(\Box^2+\mu^2\right)\phi=0 \tag{1}$$

has discrete plane-wave solutions of form

$$\phi=\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a(\mathbf{k})e^{-ikx}+b^{\dagger}(\mathbf{k})e^{ikx}\right) \tag{2}$$

and continuous plane-wave solutions of form

$$\phi(x)=\int\frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}a(\mathbf{k})e^{-ikx}+\int\frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}b^{\dagger}(\mathbf{k})e^{ikx} \tag{3}$$

The second quantization postulate is that the Poisson brackets from classical field theory translate into commutators in quantum field theory, so that

$$\left[\phi^r(\mathbf{x},t),\pi_s\left(\mathbf{x}',t\right)\right] = i\hbar\delta_s^r\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{4}$$

$$\left[\phi^r,\phi^s\right] = 0 \tag{5}$$

$$\left[\pi_r,\pi_s\right] = 0 \tag{6}$$

where the conjugate momentum is, for the free-field scalar Lagrangian density we're using here:

$$\pi_r=\frac{\partial\mathcal{L}}{\partial\dot{\phi}^r}=\dot{\phi}^{r\dagger} \tag{7}$$

In Klauber's section 3.3, he goes through a detailed calculation for the discrete solution 2 to show that applying the commutation relation 4 to this solution results in commutation relations for the a and b operators:

$$\left[a(\mathbf{k}),a^{\dagger}\left(\mathbf{k}'\right)\right]=\left[b(\mathbf{k}),b^{\dagger}\left(\mathbf{k}'\right)\right]=\delta_{\mathbf{k}\mathbf{k}'} \tag{8}$$

To abbreviate the notation a bit, we'll write this as

$$\left[a_{\mathbf{k}},a_{\mathbf{k}'}^{\dagger}\right]=\left[b_{\mathbf{k}},b_{\mathbf{k}'}^{\dagger}\right]=\delta_{\mathbf{k}\mathbf{k}'} \tag{9}$$

The derivation of the corresponding commutators for the continuous solution 3 is very similar, although a bit lengthy, but it's useful to review the process.

In the derivation, we have two fields: ϕ and $\phi^{\dagger}$ and their conjugate momenta π and $\pi^{\dagger}$. From 3 and 7, and the fact that k is a 4-vector:

$$k=\left(\omega_{\mathbf{k}},\mathbf{k}\right) \tag{10}$$

we have

$$\phi\left(\mathbf{x},t\right)=\int\frac{d^{3}k}{\sqrt{2\left(2\pi\right)^{3}\omega_{\mathbf{k}}}}a_{\mathbf{k}}e^{-i\left(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x}\right)}+\int\frac{d^{3}k}{\sqrt{2\left(2\pi\right)^{3}\omega_{\mathbf{k}}}}b_{\mathbf{k}}^{\dagger}e^{i\left(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x}\right)} \tag{11}$$

$$\pi\left(\mathbf{x}',t\right)=\int\frac{i\omega_{\mathbf{k}'}d^{3}k'}{\sqrt{2\left(2\pi\right)^{3}\omega_{\mathbf{k}'}}}a_{\mathbf{k}'}^{\dagger}e^{i\left(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x}'\right)}+\int\frac{-i\omega_{\mathbf{k}'}d^{3}k'}{\sqrt{2\left(2\pi\right)^{3}\omega_{\mathbf{k}'}}}b_{\mathbf{k}'}e^{-i\left(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x}'\right)} \tag{12}$$

I've used different integration variables k and k' for the two equations, since we'll be multiplying them and we need to keep the integration variables separate. If we substitute these two equations into the LHS of 4 and multiply them out, we get a long expression which we can write as

$$\left[\phi\left(\mathbf{x},t\right),\pi\left(\mathbf{x}',t\right)\right]=\frac{1}{2\left(2\pi\right)^{3}}\int d^{3}k\int d^{3}k'\frac{i\omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}X \tag{13}$$

where X is

$$X=\left[a_{\mathbf{k}},a_{\mathbf{k}'}^{\dagger}\right]e^{i\left(\omega_{\mathbf{k}'}-\omega_{\mathbf{k}}\right)t}e^{i\left(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}'\right)}- \tag{14}$$

$$\left[a_{\mathbf{k}},b_{\mathbf{k}'}\right]e^{-i\left(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}}\right)t}e^{i\left(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{x}'\right)}- \tag{15}$$

$$\left[a_{\mathbf{k}'}^{\dagger},b_{\mathbf{k}}^{\dagger}\right]e^{i\left(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}}\right)t}e^{-i\left(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{x}'\right)}+ \tag{16}$$

$$\left[b_{\mathbf{k}'},b_{\mathbf{k}}^{\dagger}\right]e^{-i\left(\omega_{\mathbf{k}'}-\omega_{\mathbf{k}}\right)t}e^{-i\left(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}'\right)} \tag{17}$$

The delta function on the RHS of 4 can be written as an integral:

$$\delta^{3}\left(\mathbf{x}-\mathbf{x}'\right)=\frac{1}{\left(2\pi\right)^{3}}\int d^{3}k\,e^{i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{x}'\right)} \tag{18}$$

Therefore, we must have

$$\frac{1}{2}\int d^3k'\frac{i\omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}X=e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \tag{19}$$

The RHS of 19 has no time dependence, so the terms in 15 and 16 must be zero, since they contain a factor $e^{\pm i\left(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}}\right)t}$ which is time-dependent, no matter what $\mathbf{k}$ and $\mathbf{k}'$ are (remember that $\omega_{\mathbf{k}}$ is always positive as it represents a frequency). Therefore

$$\left[a_{\mathbf{k}},b_{\mathbf{k}'}\right]=\left[a^{\dagger}_{\mathbf{k}'},b^{\dagger}_{\mathbf{k}}\right]=0 \tag{20}$$

The remaining two terms 14 and 17 are independent of time if $\omega_{\mathbf{k}}=\omega_{\mathbf{k}'}$ which implies that $|\mathbf{k}|=|\mathbf{k}'|$. Choosing

$$\left[a_{\mathbf{k}},a^{\dagger}_{\mathbf{k}'}\right]=\left[b_{\mathbf{k}'},b^{\dagger}_{\mathbf{k}}\right]=\delta^3\left(\mathbf{k}-\mathbf{k}'\right) \tag{21}$$

certainly satisfies this condition as well as the original commutator 4, since

$$\frac{1}{2(2\pi)^3}\int d^3k\int d^3k'\frac{i\omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}X=\frac{i}{2(2\pi)^3}\int d^3k\int d^3k'\delta^3\left(\mathbf{k}-\mathbf{k}'\right)\left(e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}')}+e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}')}\right) \tag{22}$$

$$=\frac{i}{2(2\pi)^3}\int d^3k\left(e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}+e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\right) \tag{23}$$

$$=i\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{24}$$

It would seem to me that the justification for this choice (rather than a more general condition which gives non-zero commutators for any pair $\mathbf{k}$ and $\mathbf{k}'$ where $|\mathbf{k}|=|\mathbf{k}'|$, and not the more restrictive condition that $\mathbf{k}=\mathbf{k}'$) is that these more general conditions don't satisfy 4.

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