VACUUM ENERGY IN THE FREE KLEIN-GORDON FIELD

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.9.

The scalar, free-field Hamiltonian we got in the last post is

$$H=\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left[N_a(\mathbf{k})+\frac{1}{2}+N_b(\mathbf{k})+\frac{1}{2}\right] \tag{1}$$

where the operators are

$$\begin{aligned} N_a(\mathbf{k}) &= a^\dagger(\mathbf{k})\,a(\mathbf{k}) && (2)\\ N_b(\mathbf{k}) &= b^\dagger(\mathbf{k})\,b(\mathbf{k}) && (3)\end{aligned}$$

The operators a and b are the coefficients in the plane-wave solution to the Klein-Gordon equation, and satisfy the commutation relations

$$\begin{aligned} \left[a_{\mathbf{k}},a_{\mathbf{k}'}^\dagger\right] &= \left[b_{\mathbf{k}},b_{\mathbf{k}'}^\dagger\right]=\delta_{\mathbf{k}\mathbf{k}'} && (4)\\ \left[a_{\mathbf{k}},b_{\mathbf{k}'}\right] &= \left[a_{\mathbf{k}},b_{\mathbf{k}'}^\dagger\right]=0 && (5)\end{aligned}$$

At this point, Klauber introduces a multi-particle quantum state. I'm not particularly clear just what form this state takes in the theory. In other words, can we write it down as some mathematical function of space and time?

As an aside at this point, you might recall that we dealt with similar operators a_- and a_+ (the lowering and raising operators) when dealing with the harmonic oscillator in non-relativistic quantum mechanics. In that case, however, these operators were defined in terms of the observable operators p (momentum) and x (position), and this allowed us to derive a differential equation for the state ψ which could be solved to give an explicit form for $\psi(x)$ for the ground state. Applying a_+ to the ground state generated successively higher excited states, which turned out to be Hermite polynomials. In the field theory case, the $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ operators appeared as coefficients in the solution of the Klein-Gordon equation, so we don't have an alternative form for them in terms of more familiar operators. As such, we can't derive a differential equation for the state $|\phi\rangle$.

In any case, suppose we have a state representing 2 free scalar particles, each with energy $\omega_{\mathbf{k}}$. We can write this state using the usual ket notation as $|2\phi_{\mathbf{k}}\rangle$. Since the energy of each particle is precisely defined, we expect this state to be an eigenstate of the Hamiltonian, with eigenvalue $2\omega_{\mathbf{k}}$, that is

$$H\,|2\omega_{\mathbf{k}}\rangle = 2\omega_{\mathbf{k}}\,|2\phi_{\mathbf{k}}\rangle \tag{6}$$

The problem with this is that the Hamiltonian 1 contains a couple of terms (the two $\frac{1}{2}$ terms) that don't depend on $\mathbf{k}$ (the $\omega_{\mathbf{k}}$ factor is just a number, not an operator, so it just multiplies any state it acts on), so these terms appear in any state, even a state with no particles at all (called the *vacuum state*). There are also two distinct operators N_a and N_b, so when we say we have a "particle", which of these operators are we referring to? The interpretation of 1 is that free scalar particles come in two varieties (particles and antiparticles), and that even the vacuum state has a non-zero energy associated with it. This interpretation works if $N_a(\mathbf{k})$ is the *number operator* for particles of type a with energy $\omega_{\mathbf{k}}$, that is, when $N_a(\mathbf{k})$ operates on the state $|n\phi_{\mathbf{k}}\rangle$, where n is an integer, it gives:

$$N_a(\mathbf{k})\,|n\phi_{\mathbf{k}}\rangle = n\,|n\phi_{\mathbf{k}}\rangle \tag{7}$$

In that case, the Hamiltonian gives

$$\begin{aligned} H\,|2\omega_{\mathbf{k}}\rangle &= 2\omega_{\mathbf{k}}\,|2\phi_{\mathbf{k}}\rangle + \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\frac{1}{2}+\frac{1}{2}\right) && (8) \\ &= 2\omega_{\mathbf{k}}\,|2\phi_{\mathbf{k}}\rangle + \sum_{\mathbf{k}} \omega_{\mathbf{k}} && (9) \end{aligned}$$

The latter sum extends over all possible energies $\omega_{\mathbf{k}}$ which could give an infinite energy if there is no upper limit to the energy of a single particle. Applying the Hamiltonian to the vacuum state $|0\rangle$ thus gives

$$H\,|0\rangle = \left[\sum_{\mathbf{k}} \omega_{\mathbf{k}}\right]|0\rangle \tag{10}$$

If these quantum states are all normalized so that $\langle 0\,|0\rangle = 1$ then we can work out the expectation value of an operator $\mathcal{O}$ in some, possibly multiparticle, state $|\phi\rangle$ in the same way as in non-relativistic quantum mechanics:

$$\langle\mathcal{O}\rangle = \langle\phi\,|\mathcal{O}|\,\phi\rangle \tag{11}$$

In particular, the *vacuum expectation value* (VEV) of the energy is

$$\langle 0|H|0\rangle = \left[\sum_{\mathbf{k}} \omega_{\mathbf{k}}\right] \langle 0|0\rangle = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \tag{12}$$

[We can take the $[\sum_{\mathbf{k}} \omega_{\mathbf{k}}]$ outside the bracket because it's just a sum of numbers, not operators.]

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