CREATION AND ANNIHILATION OPERATORS: NORMALIZATION

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.11.

The number operators are defined in terms of the creation and annihilation operators for the free scalar Hamiltonian as

$$N_a(\mathbf{k}) = a^{\dagger}(\mathbf{k})\,a(\mathbf{k}) \quad (1)$$

$$N_b(\mathbf{k}) = b^{\dagger}(\mathbf{k})\,b(\mathbf{k}) \quad (2)$$

We've seen that $a^{\dagger}(\mathbf{k})$ creates a particle of energy $\omega_{\mathbf{k}}$ when it operates on a state, and $a(\mathbf{k})$ destroys a particle with energy $\omega_{\mathbf{k}}$ when it operates on a state (if that state contains such a particle). That is, the state $a^{\dagger}(\mathbf{k})|n_{\mathbf{k}}\rangle$ is an eigenstate of $N_a(\mathbf{k})$ with eigenvalue $n_{\mathbf{k}}+1$ and $a(\mathbf{k})|n_{\mathbf{k}}\rangle$ is an eigenstate of $N_a(\mathbf{k})$ with eigenvalue $n_{\mathbf{k}}-1$. However, if we require all multiparticle states to be normalized, so that $\langle n_{\mathbf{k}}|n_{\mathbf{k}}\rangle = 1$, the states $a^{\dagger}(\mathbf{k})|n_{\mathbf{k}}\rangle$ and $a(\mathbf{k})|n_{\mathbf{k}}\rangle$ do not produce normalized states. Rather, we have

$$a^{\dagger}(\mathbf{k})|n_{\mathbf{k}}\rangle = A|n_{\mathbf{k}}+1\rangle \quad (3)$$

$$a(\mathbf{k})|n_{\mathbf{k}}\rangle = B|n_{\mathbf{k}}-1\rangle \quad (4)$$

for some constants A and B that are determined by requiring normalization.

To find A and B, we can take the modulus of the states above. We get (we'll leave off the $(\mathbf{k})$ dependence of the $a^{\dagger}$ and a operators to save typing; everything in what follows occurs at wave number $\mathbf{k}$; we'll also assume A and B are real, although they could also be multiplied by some phase factor $e^{i\alpha}$, but this just complicates things unnecessarily). By using the commutation relaton

$$\left[a,a^{\dagger}\right] = 1 \quad (5)$$

we get, from 3

$$\begin{aligned}
\left\langle n_{\mathbf{k}}\left|aa^{\dagger}\right|n_{\mathbf{k}}\right\rangle &= A^{2}\left\langle n_{\mathbf{k}}+1\left|n_{\mathbf{k}}+1\right.\right\rangle = A^{2} && (6)\\
\left\langle n_{\mathbf{k}}\left|aa^{\dagger}\right|n_{\mathbf{k}}\right\rangle &= \left\langle n_{\mathbf{k}}\left|a^{\dagger}a+1\right|n_{\mathbf{k}}\right\rangle && (7)\\
&= \left\langle n_{\mathbf{k}}\left|N_{a}+1\right|n_{\mathbf{k}}\right\rangle && (8)\\
&= \left(n_{\mathbf{k}}+1\right)\left\langle n_{\mathbf{k}}\left|n_{\mathbf{k}}\right.\right\rangle && (9)\\
&= \left(n_{\mathbf{k}}+1\right) && (10)\\
A &= \sqrt{n_{\mathbf{k}}+1} && (11)
\end{aligned}$$

Therefore

$$a^{\dagger}(\mathbf{k})\left|n_{\mathbf{k}}\right\rangle = \sqrt{n_{\mathbf{k}}+1}\left|n_{\mathbf{k}}+1\right\rangle \tag{12}$$

For the annihilation operator, we have from 4:

$$\begin{aligned}
\left\langle n_{\mathbf{k}}\left|a^{\dagger}a\right|n_{\mathbf{k}}\right\rangle &= B^{2}\left\langle n_{\mathbf{k}}-1\left|n_{\mathbf{k}}-1\right.\right\rangle = B^{2} && (13)\\
\left\langle n_{\mathbf{k}}\left|a^{\dagger}a\right|n_{\mathbf{k}}\right\rangle &= \left\langle n_{\mathbf{k}}\left|N_{a}\right|n_{\mathbf{k}}\right\rangle && (14)\\
&= n_{\mathbf{k}}\left\langle n_{\mathbf{k}}\left|n_{\mathbf{k}}\right.\right\rangle && (15)\\
&= n_{\mathbf{k}} && (16)\\
B &= \sqrt{n_{\mathbf{k}}} && (17)
\end{aligned}$$

Therefore

$$a(\mathbf{k})\left|n_{\mathbf{k}}\right\rangle = \sqrt{n_{\mathbf{k}}}\left|n_{\mathbf{k}}-1\right\rangle \tag{18}$$

This relation implies that applying $a(\mathbf{k})$ to a state that contains no particles with energy $\omega_{\mathbf{k}}$ (that is, where $n_{\mathbf{k}}=0$) produces 0. In particular, if we apply $a(\mathbf{k})$ to the vacuum state, we end up with no state at all:

$$a(\mathbf{k})\left|0\right\rangle = 0 \tag{19}$$

Note that $\left|0\right\rangle$ and 0 aren't the same thing: $\left|0\right\rangle$ is a quantum state with no particles in it, while 0 is mathematically zero, that is, nothing. As an analogy, $\left|0\right\rangle$ is like having a bucket with nothing in it, while 0 corresponds to removing the bucket as well.

We can repeat exactly the same calculations for the antiparticle operators $b^{\dagger}$ and b and get the results

$$\begin{aligned}
b^{\dagger}(\mathbf{k})\left|\bar{n}_{\mathbf{k}}\right\rangle &= \sqrt{\bar{n}_{\mathbf{k}}+1}\left|\bar{n}_{\mathbf{k}}+1\right\rangle && (20)\\
b(\mathbf{k})\left|\bar{n}_{\mathbf{k}}\right\rangle &= \sqrt{\bar{n}_{\mathbf{k}}}\left|\bar{n}_{\mathbf{k}}-1\right\rangle && (21)
\end{aligned}$$

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