PROBABILITY DENSITY IN A KLEIN-GORDON FIELD

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.12.

The probability density ρ and current $\mathbf{j}$ for the solutions ϕ to the Klein-Gordon equation in relativistic quantum mechanics (where ϕ represents a state, not a field) are

$$\rho \equiv i\left(\phi^{\dagger}\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^{\dagger}}{\partial t}\right) \tag{1}$$

$$\mathbf{j} = -i\left(\phi^{\dagger}\nabla\phi-\phi\nabla\phi^{\dagger}\right) \tag{2}$$

In field theory, the solutions to the Klein-Gordon equation are mathematically the same as in relativistic quantum mechanics; the only difference is that the coefficients are creation and annihilation operators rather than just numbers. The field solutions are

$$\phi(x)=\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}a(\mathbf{k})e^{-ikx}+\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}b^{\dagger}(\mathbf{k})e^{ikx} \tag{3}$$

$$\equiv\phi^{+}+\phi^{-} \tag{4}$$

$$\phi^{\dagger}(x)=\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}a^{\dagger}(\mathbf{k})e^{ikx}+\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}b(\mathbf{k})e^{-ikx} \tag{5}$$

$$\equiv\phi^{\dagger+}+\phi^{\dagger-} \tag{6}$$

Since the solutions are mathematically the same in the two cases, the derivation of 1 and 2 is the same. To work out the densities, we must use the field solutions instead of the state solutions.

To work out the probability density ρ for a given state $|\phi_1,\phi_2,\phi_3,\ldots\rangle$ we need to evaluate the expression

$$\langle\phi_1,\phi_2,\phi_3,\ldots|\rho|\phi_1,\phi_2,\phi_3,\ldots\rangle \tag{7}$$

Looking at the fields 3 and 5, we see that their time derivatives will bring down a factor of $\pm i\omega_{\mathbf{k}}$ but otherwise leave the expressions unchanged, so the terms on the RHS of 1 will involve products of two of the operators $a^{\dagger}$, a, $b^{\dagger}$ and b. Since the state in the bra of 7 is the same as the state in the ket, only combinations of these operators that leave the ket state unchanged will survive the calculation (due to the orthogonality of states with different numbers of particles or different energies $\omega_{\mathbf{k}}$ in them). This means that all terms containing a product of $a^{\dagger}$ or a with $b^{\dagger}$ or b will contribute nothing, since they don't leave the ket unchanged. We can therefore look at a and b separately and then combine the result.

First, we'll look at the b terms. We can use the earlier result with the original A and B coefficients replaced by a and b operators. This gives (I've left off the bra and ket from 7 to make the typesetting easier, but you should imagine both sides enclosed within this bra and ket):

$$\begin{aligned}\rho_b &= -\left[\sum_{\mathbf{k}}\frac{b_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}}e^{-ikx}\right]\left[\sum_{\mathbf{k}'}\frac{\omega_{\mathbf{k}'}b_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}}e^{ik'x}\right]- \\ &\quad \left[\sum_{\mathbf{k}'}\frac{b_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}}e^{ik'x}\right]\left[\sum_{\mathbf{k}}\frac{\omega_{\mathbf{k}}b_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}}e^{-ikx}\right] \qquad (8)\end{aligned}$$

Only terms in the double sums where $\mathbf{k}=\mathbf{k}'$ will survive, again because of the orthogonality of state with different $\mathbf{k}$ values. Therefore we have

$$\begin{aligned}\rho_b &= -\frac{1}{2V}\sum_{\mathbf{k}}\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) &(9)\\ &= -\frac{1}{2V}\sum_{\mathbf{k}}\left(1+b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) &(10)\\ &= -\frac{1}{V}\sum_{\mathbf{k}}\left(\frac{1}{2}+b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) &(11)\\ &= -\frac{1}{V}\sum_{\mathbf{k}}\left(\frac{1}{2}+N_b(\mathbf{k})\right) &(12)\end{aligned}$$

where we used the commutator $\left[b_{\mathbf{k}},b_{\mathbf{k}}^{\dagger}\right]=1$ in the second line.

For the a operators we have

$$\begin{aligned}
\rho_a &= \left[\sum_{\mathbf{k}} \frac{a_{\mathbf{k}}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}} V}} e^{ikx}\right]\left[\sum_{\mathbf{k}'} \frac{\omega_{\mathbf{k}'} a_{\mathbf{k}'}}{\sqrt{2\omega_{\mathbf{k}'} V}} e^{-ik'x}\right] + \\
&\quad \left[\sum_{\mathbf{k}'} \frac{a_{\mathbf{k}'}}{\sqrt{2\omega_{\mathbf{k}'} V}} e^{-ik'x}\right]\left[\sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}} a_{\mathbf{k}}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}} V}} e^{ikx}\right] && (13) \\
&= \frac{1}{2V} \sum_{\mathbf{k}} \left(a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + a_{\mathbf{k}} a_{\mathbf{k}}^{\dagger}\right) && (14) \\
&= \frac{1}{2V} \sum_{\mathbf{k}} \left(a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + 1 + a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}\right) && (15) \\
&= \frac{1}{V} \sum_{\mathbf{k}} \left(\frac{1}{2} + a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}\right) && (16) \\
&= \frac{1}{V} \sum_{\mathbf{k}} \left(\frac{1}{2} + N_a(\mathbf{k})\right) && (17)
\end{aligned}$$

The total probability density for the state $|\phi_1, \phi_2, \phi_3, \ldots\rangle$ is therefore

$$\begin{aligned}
\bar{\rho} &= \langle \phi_1, \phi_2, \phi_3, \ldots | \rho | \phi_1, \phi_2, \phi_3, \ldots \rangle && (18) \\
&= \rho_a + \rho_b && (19) \\
&= \frac{1}{V} \sum_{\mathbf{k}} \left(N_a(\mathbf{k}) - N_b(\mathbf{k})\right) && (20)
\end{aligned}$$

The probability density is seen to be the number density (number of particles per unit volume), except that b-particles (antiparticles) count as negative particles.

It should be noted that we've implicitly assumed that the state $|\phi_1, \phi_2, \phi_3, \ldots\rangle$ consists entirely of particles that are in energy eigenstates, that is the energy of each particle is precisely $\omega_{\mathbf{k}}$ for some $\mathbf{k}$ (though not all particles need be in the same eigenstate). If this weren't the case, the actions of the operators $a^{\dagger}$, a, $b^{\dagger}$ and b aren't quite so straightforward and we couldn't get the simple result we did.