# KLEIN-GORDON EQUATION WITH ANTICOMMUTATORS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 3, Problem 3.16.

The derivation of the commutation relations for the creation and annihilation operators used the requirement that the fields and conjugate momentum densities obeyed the commutation relations translated from classical physics, namely

$$\left[\phi^{r}(\mathbf{x},t),\pi_{s}\left(\mathbf{x}',t\right)\right] = i\hbar\delta_{s}^{r}\delta^{3}\left(\mathbf{x}-\mathbf{x}'\right) \tag{1}$$

$$\left[\phi^{r},\phi^{s}\right] = 0 \tag{2}$$

$$\left[\pi_{r},\pi_{s}\right] = 0 \tag{3}$$

If these relations had instead been *anti*commutators, we can derive anticommutators for the creation and annihilation operators in much the same way. That is, we start by requiring

$$\left\{\phi^{r}(\mathbf{x},t),\pi_{s}\left(\mathbf{x}',t\right)\right\} = i\hbar\delta_{s}^{r}\delta^{3}\left(\mathbf{x}-\mathbf{x}'\right) \tag{4}$$

$$\left\{\phi^{r},\phi^{s}\right\} = 0 \tag{5}$$

$$\left\{\pi_{r},\pi_{s}\right\} = 0 \tag{6}$$

where the braces indicate an anticommutator:

$$\{A,B\} \equiv AB+BA \tag{7}$$

[There are several notations in use for anticommutators. Klauber uses $[A,B]_{+}$ instead of $\{A,B\}$.]

Klauber's problem 3.16 asks us to go through the derivation for the discrete case, but since I've already done the commutator derivation for the continuous case and the two cases are very similar, I'll run through the anticommutator derivation for the continuous case to save myself a lot of typing.

We start with the continuous plane-wave solutions of form

$$\phi(x) = \int\frac{d^{3}k}{\sqrt{2(2\pi)^{3}\omega_{\mathbf{k}}}}a(\mathbf{k})e^{-ikx}+\int\frac{d^{3}k}{\sqrt{2(2\pi)^{3}\omega_{\mathbf{k}}}}b^{\dagger}(\mathbf{k})e^{ikx} \tag{8}$$

The conjugate momentum is, for the free-field scalar Lagrangian density we're using here:

$$\pi_r = \frac{\partial \mathscr{L}}{\partial \dot{\phi}^r} = \dot{\phi}^{r\dagger} \tag{9}$$

In the derivation, we have two fields: $\phi$ and $\phi^\dagger$ and their conjugate momenta $\pi$ and $\pi^\dagger$. From 8 and 9, and the fact that $k$ is a 4-vector:

$$k = (\omega_{\mathbf{k}}, \mathbf{k}) \tag{10}$$

we have

(11)
$$\phi(\mathbf{x},t) = \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} a_{\mathbf{k}} e^{-i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} + \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} b_{\mathbf{k}}^\dagger e^{i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})}$$

(12)
$$\pi(\mathbf{x}',t) = \int \frac{i\omega_{\mathbf{k}'} d^3k'}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}'}}} a_{\mathbf{k}'}^\dagger e^{i\left(\omega_{\mathbf{k}'}t - \mathbf{k}'\cdot\mathbf{x}'\right)} + \int \frac{-i\omega_{\mathbf{k}'} d^3k'}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}'}}} b_{\mathbf{k}'} e^{-i\left(\omega_{\mathbf{k}'}t - \mathbf{k}'\cdot\mathbf{x}'\right)}$$

I've used different integration variables $k$ and $k'$ for the two equations, since we'll be multiplying them and we need to keep the integration variables separate. If we substitute these two equations into the LHS of 4 and multiply them out, we get a long expression which we can write as

$$\left\{\phi(\mathbf{x},t), \pi(\mathbf{x}',t)\right\} = \frac{1}{2(2\pi)^3} \int d^3k \int d^3k' \frac{i\omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} X \tag{13}$$

where $X$ is

$$\begin{aligned}
X \quad = \quad & \left\{a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger\right\} e^{i\left(\omega_{\mathbf{k}'} - \omega_{\mathbf{k}}\right)t} e^{i(\mathbf{k}\cdot\mathbf{x} - \mathbf{k}'\cdot\mathbf{x}')} - && (14)\\
& \left\{a_{\mathbf{k}}, b_{\mathbf{k}'}\right\} e^{-i\left(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}}\right)t} e^{i(\mathbf{k}\cdot\mathbf{x} + \mathbf{k}'\cdot\mathbf{x}')} + && (15)\\
& \left\{a_{\mathbf{k}'}^\dagger, b_{\mathbf{k}}^\dagger\right\} e^{i\left(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}}\right)t} e^{-i(\mathbf{k}\cdot\mathbf{x} + \mathbf{k}'\cdot\mathbf{x}')} - && (16)\\
& \left\{b_{\mathbf{k}'}, b_{\mathbf{k}}^\dagger\right\} e^{-i\left(\omega_{\mathbf{k}'} - \omega_{\mathbf{k}}\right)t} e^{-i(\mathbf{k}\cdot\mathbf{x} - \mathbf{k}'\cdot\mathbf{x}')} && (17)
\end{aligned}$$

[Note that the pattern of + and $-$ signs on these terms is different from the commutator case because anticommutators are symmetric, that is $\{A,B\} = \{B,A\}$ while commutators are antisymmetric: $[A,B] = -[B,A]$.]

The delta function on the RHS of 4 can be written as an integral:

$$\delta^3\left(\mathbf{x}-\mathbf{x}'\right)=\frac{1}{(2\pi)^3}\int d^3k\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \tag{18}$$

Therefore, we must have

$$\frac{1}{2}\int d^3k'\frac{i\omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}X=e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \tag{19}$$

The RHS of 19 has no time dependence, so the terms in 15 and 16 must be zero, since they contain a factor $e^{\pm i\left(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}}\right)t}$ which is time-dependent, no matter what $\mathbf{k}$ and $\mathbf{k}'$ are (remember that $\omega_{\mathbf{k}}$ is always positive as it represents a frequency). Therefore

$$\{a_{\mathbf{k}},b_{\mathbf{k}'}\}=\left\{a^\dagger_{\mathbf{k}'},b^\dagger_{\mathbf{k}}\right\}=0 \tag{20}$$

The remaining two terms 14 and 17 are independent of time if $\omega_{\mathbf{k}}=\omega_{\mathbf{k}'}$ which implies that $|\mathbf{k}|=|\mathbf{k}'|$. Choosing

$$\left\{a_{\mathbf{k}},a^\dagger_{\mathbf{k}'}\right\}=-\left\{b_{\mathbf{k}'},b^\dagger_{\mathbf{k}}\right\}=\delta^3\left(\mathbf{k}-\mathbf{k}'\right) \tag{21}$$

certainly satisfies this condition as well as the original commutator 4, since

$$\frac{1}{2(2\pi)^3}\int d^3k\int d^3k'\frac{i\omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}X=\frac{i}{2(2\pi)^3}\int d^3k\int d^3k'\,\delta^3\left(\mathbf{k}-\mathbf{k}'\right)\left(e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}')}+e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}')}\right) \tag{22}$$

$$=\frac{i}{2(2\pi)^3}\int d^3k\left(e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}+e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\right) \tag{23}$$

$$=i\delta^3\left(\mathbf{x}-\mathbf{x}'\right) \tag{24}$$