DIRAC EQUATION: MATRIX PROPERTIES

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.1.

In our earlier look at the Dirac equation, we simply stated the equation and then derived a few properties of its components. It seems that this is pretty much the way Dirac himself derived it. He was seeking a relativistic generalization of the Schrödinger equation, so he wanted his equation to have the form

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \tag{1}$$

for some Hamiltonian H. The original Schrödinger equation defined H (for a free particle) as $H = -\frac{\nabla^2}{2m}$, so this non-relativistic equation had a first-order time derivative and a second order spatial derivative. The Klein-Gordon equation contains second-order derivatives in both time and space.

The problem Dirac faced was therefore how to define H to make 1 relativistic and at the same time, retain the first order time derivative. In special relativity, energy, mass and momentum are related by

$$E^2 = p^2 + m^2 \tag{2}$$

If we use the operator form for the momentum: $p^i = -i\partial^i$, then attempting to use $E = \sqrt{p^2+m^2}$ as the Hamiltonian operator runs into the problem that the momentum operator is inside a square root. Dirac solved this problem by proposing that

$$H = \alpha\cdot\mathbf{p} + \beta m \tag{3}$$

where α is a 3-d vector of matrices and β is a single (scalar) matrix. We can then require that H^2 give the energy squared as specified in 2, which in turn imposes conditions on the matrices α and β. We saw in the earlier post that this leads to the conditions that all four matrices have zero trace, have ±1 eigenvalues, and have even dimension, with the minimum dimension being 4×4. The matrices also satisfy anticommutation relations:

$$\begin{aligned} \{\alpha_i,\alpha_j\} &= \{\alpha_i,\beta\} = 0 \text{ if } i \neq j && (4)\\ \alpha_i^2 = \beta^2 &= I && (5) \end{aligned}$$

Dirac and Pauli found that the smallest matrices that satisfy all these conditions are

$$\alpha_1 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \tag{6}$$

$$\alpha_2 = \begin{bmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{bmatrix} \tag{7}$$

$$\alpha_3 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{bmatrix} \tag{8}$$

$$\beta = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \tag{9}$$

where all blank matrix elements are zero. These can be written in a more condensed form by using the Pauli spin matrices from non-relativistic quantum mechanics

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{10}$$

We get

$$\alpha_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \tag{11}$$

$$\alpha_2 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} \tag{12}$$

$$\alpha_3 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix} \tag{13}$$

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{14}$$

Here, each entry in each matrix is a 2×2 submatrix given by 10.

It's fairly obvious from their definitions that all 4 matrices have zero trace and are hermitian (that is, they equal their complex conjugate transpose).

The eigenvalues of β are ±1 since the matrix is diagonal. The eigenvalues of the α_i can be found in the usual way. For α_1 we get

$$\det\left(\alpha_1-\lambda I\right)=\left|\begin{array}{cccc}-\lambda & 0 & 0 & 1\\ 0 & -\lambda & 1 & 0\\ 0 & 1 & -\lambda & 0\\ 1 & 0 & 0 & -\lambda\end{array}\right| \tag{15}$$

$$=-\lambda\left[-\lambda\left(\lambda^2\right)-(-\lambda)\right]-\left(\lambda^2-1\right) \tag{16}$$

$$=\lambda^4-2\lambda^2+1 \tag{17}$$

$$=\left(\lambda^2-1\right)^2 \tag{18}$$

The roots of this equation are ±1 (twice each). The eigenvalues of α_2 and α_3 also turn out to be ±1 as can be verified if you grind through the calculations. The conditions 4 and 5 can be verified by direct multiplication (although this is kind of tedious).

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