DIRAC EQUATION AS FOUR COUPLED DIFFERENTIAL EQUATIONS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.4.

The Dirac equation in relativistic quantum mechanics can be written as

$$i\frac{\partial}{\partial t}\left|\psi\right\rangle=\left(\alpha\cdot\mathbf{p}+\beta m\right)\left|\psi\right\rangle \tag{1}$$

where the matrices are given by

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

$$\alpha_2 = \begin{bmatrix} 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ 0 & -i & 0 & 0\\ i & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

$$\alpha_3 = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{4}$$

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{5}$$

The Dirac equation is usually written in terms of the γ^{μ} matrices, defined as

$$\gamma^0 = \beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{6}$$

$$\gamma^1 = \beta\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

$$\gamma^2 = \beta\alpha_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

$$\gamma^3 = \beta\alpha_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{9}$$

If we multiply 1 on both sides on the left with $\gamma^0 = \beta$ and use $\beta^2 = I$, where I is the identity matrix, we get

$$i\gamma^0 \frac{\partial}{\partial t}|\psi\rangle = \left(\gamma \cdot \mathbf{p} + \beta^2 m\right)|\psi\rangle \tag{10}$$

$$i\gamma^0 \partial_0 |\psi\rangle = (\gamma \cdot \mathbf{p} + mI)|\psi\rangle \tag{11}$$

The momentum **p** is, in operator form

$$p_i = -i\partial_i \tag{12}$$

so we can shift everything to the LHS and get

$$(i\gamma^\mu \partial_\mu - mI)|\psi\rangle = 0 \tag{13}$$

Note that the index μ on γ^μ is *not* a spacetime index. Rather it is an index in spinor space (although we haven't actually connected the Dirac equation to spin yet, so you'll have to take it on faith at this point). In non-relativistic quantum mechanics, the spin component of a spin-$\frac{1}{2}$ particle is represented by a 2×2 matrix which is in a separate space (2-d spinor space) from the 3-d position space. To satisfy the anticommutation conditions, we saw that the Dirac equation matrices must have an even dimension of at least 4×4, and it turns out that 2 of these dimensions refer to spin up and spin down for a spin $\frac{1}{2}$ particle, while the other 2 dimensions refer to spin up and spin down for a spin $\frac{1}{2}$ antiparticle. More on this in future posts (hopefully).

Returning to 13, since the matrices on the LHS are 4×4, the state ket $|\psi\rangle$ must be a 4-d vector, so that 13 actually represents four separate equations. That is, the equation actually states that

$$\left(i\gamma^{\mu}\partial_{\mu}-mI\right)\left[\begin{array}{c}|\psi\rangle_{1}\\|\psi\rangle_{2}\\|\psi\rangle_{3}\\|\psi\rangle_{4}\end{array}\right]=\left[\begin{array}{c}0\\0\\0\\0\end{array}\right] \tag{14}$$

The indices here can be a bit confusing. The index μ runs from 0 to 3, while the index on the kets $|\psi\rangle_{\eta}$ runs from 1 to 4. To help keep things straight, remember that μ runs over the spacetime coordinates where time has index $\mu=0$ and space covers $\mu=1,2,3$, while η runs over components in spinor space, where the indices run from 1 to 4. That is, η refers to the columns within the matrix $i\gamma^{\mu}\partial_{\mu}-mI$ as they multiply into the column vector $|\psi\rangle$.

Written out in terms of components, we have

$$\sum_{\eta=1}^{4}\left[\sum_{\mu=0}^{3}i\left(\gamma^{\mu}\right)_{\kappa\eta}\partial_{\mu}-m\delta_{\kappa\eta}\right]|\psi\rangle_{\eta}=0 \tag{15}$$

where $\kappa=1,2,3,4$ labels which row in the matrix $i\gamma^{\mu}\partial_{\mu}-mI$ is being multiplied into $|\psi\rangle$.

To understand this equation, we can look at each κ separately. For $\kappa=1$ we have (referring back to the γ^{μ} matrices above):

$$\begin{aligned}\sum_{\eta=1}^{4}\left[\sum_{\mu=0}^{3}i\left(\gamma^{\mu}\right)_{1\eta}\partial_{\mu}-m\delta_{1\eta}\right]|\psi\rangle_{\eta} &= \left(i\gamma_{11}^{0}\partial_{0}-m\right)|\psi\rangle_{1}+i\gamma_{13}^{3}\partial_{3}|\psi\rangle_{3}+\\ &\quad i\left(\gamma_{14}^{1}\partial_{1}+\gamma_{14}^{2}\partial_{2}\right)|\psi\rangle_{4} &(16)\\ &= \left(i\partial_{0}-m\right)|\psi\rangle_{1}+i\partial_{3}|\psi\rangle_{3}+\\ &\quad \left(i\partial_{1}+\partial_{2}\right)|\psi\rangle_{4} &(17)\end{aligned}$$

Thus $\kappa=1$ corresponds to the differential equation

$$\left(i\partial_{0}-m\right)|\psi\rangle_{1}+i\partial_{3}|\psi\rangle_{3}+\left(i\partial_{1}+\partial_{2}\right)|\psi\rangle_{4}=0 \tag{18}$$

For $\kappa=2$ we get

$$\sum_{\eta=1}^{4}\left[\sum_{\mu=0}^{3} i\left(\gamma^{\mu}\right)_{2\eta}\partial_{\mu} - m\delta_{2\eta}\right]|\psi\rangle_{\eta} = \left(i\gamma_{22}^{0}\partial_{0} - m\right)|\psi\rangle_{2} + i\left(\gamma_{23}^{1}\partial_{1} + \gamma_{23}^{2}\partial_{2}\right)|\psi\rangle_{3} + i\gamma_{24}^{3}\partial_{3}|\psi\rangle_{4} \tag{19}$$

$$= \left(i\partial_{0} - m\right)|\psi\rangle_{2} + \left(i\partial_{1} - \partial_{2}\right)|\psi\rangle_{3} - i\partial_{3}|\psi\rangle_{4} \tag{20}$$

Thus $\kappa = 2$ corresponds to the differential equation

$$\left(i\partial_{0} - m\right)|\psi\rangle_{2} + \left(i\partial_{1} - \partial_{2}\right)|\psi\rangle_{3} - i\partial_{3}|\psi\rangle_{4} = 0 \tag{21}$$

For $\kappa = 3$ we get

$$\sum_{\eta=1}^{4}\left[\sum_{\mu=0}^{3} i\left(\gamma^{\mu}\right)_{3\eta}\partial_{\mu} - m\delta_{3\eta}\right]|\psi\rangle_{\eta} = i\gamma_{31}^{3}\partial_{3}|\psi\rangle_{1} + i\left(\gamma_{32}^{1}\partial_{1} + \gamma_{32}^{2}\partial_{2}\right)|\psi\rangle_{2} + \left(i\gamma_{33}^{0}\partial_{0} - m\right)|\psi\rangle_{3} \tag{22}$$

$$= -i\partial_{3}|\psi\rangle_{1} + \left(-i\partial_{1} - \partial_{2}\right)|\psi\rangle_{2} - \left(i\partial_{0} + m\right)|\psi\rangle_{3} \tag{23}$$

Thus $\kappa = 3$ corresponds to the differential equation

$$-i\partial_{3}|\psi\rangle_{1} - \left(i\partial_{1} + \partial_{2}\right)|\psi\rangle_{2} - \left(i\partial_{0} + m\right)|\psi\rangle_{3} = 0 \tag{24}$$

Finally for $\kappa = 4$ we get

$$\sum_{\eta=1}^{4}\left[\sum_{\mu=0}^{3} i\left(\gamma^{\mu}\right)_{4\eta}\partial_{\mu} - m\delta_{4\eta}\right]|\psi\rangle_{\eta} = i\left(\gamma_{41}^{1}\partial_{1} + \gamma_{41}^{2}\partial_{2}\right)|\psi\rangle_{1} + i\gamma_{42}^{3}\partial_{3}|\psi\rangle_{2} + \left(i\gamma_{44}^{0}\partial_{0} - m\right)|\psi\rangle_{4} \tag{25}$$

$$= i\left(-\partial_{1} - i\partial_{2}\right)|\psi\rangle_{1} + i\partial_{3}|\psi\rangle_{2} + \left(-i\partial_{0} - m\right)|\psi\rangle_{4} \tag{26}$$

Thus $\kappa = 4$ corresponds to the differential equation

$$-i\left(\partial_{1} + i\partial_{2}\right)|\psi\rangle_{1} + i\partial_{3}|\psi\rangle_{2} - \left(i\partial_{0} + m\right)|\psi\rangle_{4} = 0 \tag{27}$$

The four coupled PDEs 18, 21, 24 and 27 together constitute the Dirac equation written out explicitly in terms of the matrix components.

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