DIRAC EQUATION: 4 SOLUTION VECTORS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.5.

The Dirac equation in relativistic quantum mechanics can be written as

$$\left(i\gamma^{\mu}\partial_{\mu}-mI\right)|\psi\rangle=0 \tag{1}$$

When written out in its matrix components, this equation is actually 4 differential equations.

$$\left(i\partial_0-m\right)|\psi\rangle_1+i\partial_3|\psi\rangle_3+\left(i\partial_1+\partial_2\right)|\psi\rangle_4=0 \tag{2}$$

$$\left(i\partial_0-m\right)|\psi\rangle_2+\left(i\partial_1-\partial_2\right)|\psi\rangle_3-i\partial_3|\psi\rangle_4=0 \tag{3}$$

$$-i\partial_3|\psi\rangle_1-\left(i\partial_1+\partial_2\right)|\psi\rangle_2-\left(i\partial_0+m\right)|\psi\rangle_3=0 \tag{4}$$

$$-i\left(\partial_1+i\partial_2\right)|\psi\rangle_1+i\partial_3|\psi\rangle_2-\left(i\partial_0+m\right)|\psi\rangle_4=0 \tag{5}$$

Remember that $|\psi\rangle$ is a 4-d column vector in spinor space rather than a single function, so that the subscript index j in $|\psi\rangle_j$ indicates which component in spinor space we're dealing with. These equations have four solutions denoted by $\left|\psi^{(n)}\right\rangle$ for $n=1,2,3,4$. Note that each $\left|\psi^{(n)}\right\rangle$ is a full 4-component vector in spinor space; that is, the superscript (n) indicates which complete vector we're dealing with. Thus $\left|\psi^{(n)}\right\rangle_j$ is the jth component of the nth vector.

We can write the 4 PDEs as a matrix eigenvalue equation by moving the terms involving m to the RHS and factoring out an i from the terms remaining on the LHS:

$$i\left[\begin{array}{cccc}\partial_0 & 0 & \partial_3 & \partial_1-i\partial_2\\ 0 & \partial_0 & \partial_1+i\partial_2 & -\partial_3\\ -\partial_3 & -\partial_1+i\partial_2 & -\partial_0 & 0\\ -\partial_1-i\partial_2 & \partial_3 & 0 & -\partial_0\end{array}\right]\left[\begin{array}{c}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{array}\right]=m\left[\begin{array}{c}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{array}\right] \tag{6}$$

We'll now look at the four solutions $\left|\psi^{(n)}\right\rangle$ and verify that they satisfy 6. First, we have

$$\left|\psi^{(1)}\right\rangle=\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c}1\\0\\\frac{p^{3}}{E+m}\\\frac{p^{1}+ip^{2}}{E+m}\end{array}\right]e^{-ipx}\equiv u_{1}e^{-ipx}\tag{7}$$

where u_1 is defined by this equation as the constant $\sqrt{\frac{E+m}{2m}}$ multiplied by the 4-d spinor factor. Remember that px is a 4-vector product:

$$px=p^{\mu}x_{\mu}=Et-\mathbf{p}\cdot\mathbf{x}\tag{8}$$

The derivatives in 6 are all with respect to spacetime variables, so act only on e^{-ipx}; the spinor components are constants with respect to these derivatives. The first row in 6 is therefore

$$i\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[-iE+0+\frac{p^{3}}{E+m}\left(ip^{3}\right)+\frac{p^{1}+ip^{2}}{E+m}\left(ip^{1}+p^{2}\right)\right]=\tag{9}$$

$$-\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[-E+\frac{\mathbf{p}^{2}}{E+m}\right]=\tag{10}$$

$$-\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[-E+\frac{E^{2}-m^{2}}{E+m}\right]=\tag{11}$$

$$-\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[-E+\frac{(E+m)(E-m)}{E+m}\right]=\sqrt{\frac{E+m}{2m}}e^{-ipx}m\tag{12}$$

$$=m\psi_{1}\tag{13}$$

Thus the first row of 6 is verified. The other 3 rows can be verified similarly. For row 2:

$$i\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[0+0+\frac{p^{3}}{E+m}\left(ip^{1}-p^{2}\right)+\frac{p^{1}+ip^{2}}{E+m}\left(-ip^{3}\right)\right]=0=m\psi_{2}\tag{14}$$

For row 3:

$$i\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[-ip^{3}+0+\frac{p^{3}}{E+m}(iE)+0\right]=i\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[\frac{-ip^{3}(E+m)+ip^{3}E}{E+m}\right] \tag{15}$$

$$=\sqrt{\frac{E+m}{2m}}e^{-ipx}m\frac{p^{3}}{E+m} \tag{16}$$

$$=m\psi_{3} \tag{17}$$

And for row 4:

$$i\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[\left(-ip^{1}+p^{2}\right)+0+0+\frac{p^{1}+ip^{2}}{E+m}iE\right]= \tag{18}$$

$$\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[\left(p^{1}+ip^{2}\right)-\frac{p^{1}+ip^{2}}{E+m}E\right]= \tag{19}$$

$$\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[\frac{\left(p^{1}+ip^{2}\right)(E+m)-\left(p^{1}+ip^{2}\right)E}{E+m}\right]= \tag{20}$$

$$\sqrt{\frac{E+m}{2m}}e^{-ipx}\frac{p^{1}+ip^{2}}{E+m}m=m\psi_{4} \tag{21}$$

The other 3 solutions are

$$\left|\psi^{(2)}\right\rangle=\sqrt{\frac{E+m}{2m}}\begin{bmatrix}0\\1\\\frac{p^{1}-ip^{2}}{E+m}\\-\frac{p^{3}}{E+m}\end{bmatrix}e^{-ipx}\equiv u_{2}e^{-ipx} \tag{22}$$

$$\left|\psi^{(3)}\right\rangle=\sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^{3}}{E+m}\\\frac{p^{1}+ip^{2}}{E+m}\\1\\0\end{bmatrix}e^{ipx}\equiv v_{2}e^{ipx} \tag{23}$$

$$\left|\psi^{(4)}\right\rangle=\sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^{1}-ip^{2}}{E+m}\\-\frac{p^{3}}{E+m}\\0\\1\end{bmatrix}e^{ipx}\equiv v_{1}e^{ipx} \tag{24}$$

If you really want to, you can verify that these 3 vectors satisfy 6 by grinding through the calculations as above. One point worth noting is that the constant $\sqrt{\frac{E+m}{2m}}$ that multiplies all the solutions could be any other constant and still satisfy 6 (since the constant just cancels off both sides). It's chosen to be $\sqrt{\frac{E+m}{2m}}$ to make later calculations easier.

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