

## DIRAC EQUATION: 4 SOLUTION VECTORS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.5.

The Dirac equation in relativistic quantum mechanics can be written as

$$(i\gamma^\mu \partial_\mu - mI) |\psi\rangle = 0 \quad (1)$$

When written out in its matrix components, this equation is actually 4 differential equations.

$$(i\partial_0 - m) |\psi\rangle_1 + i\partial_3 |\psi\rangle_3 + (i\partial_1 + \partial_2) |\psi\rangle_4 = 0 \quad (2)$$

$$(i\partial_0 - m) |\psi\rangle_2 + (i\partial_1 - \partial_2) |\psi\rangle_3 - i\partial_3 |\psi\rangle_4 = 0 \quad (3)$$

$$-i\partial_3 |\psi\rangle_1 - (i\partial_1 + \partial_2) |\psi\rangle_2 - (i\partial_0 + m) |\psi\rangle_3 = 0 \quad (4)$$

$$-i(\partial_1 + i\partial_2) |\psi\rangle_1 + i\partial_3 |\psi\rangle_2 - (i\partial_0 + m) |\psi\rangle_4 = 0 \quad (5)$$

Remember that  $|\psi\rangle$  is a 4-d column vector in spinor space rather than a single function, so that the subscript index  $j$  in  $|\psi\rangle_j$  indicates which component in spinor space we're dealing with. These equations have four solutions denoted by  $|\psi^{(n)}\rangle$  for  $n = 1, 2, 3, 4$ . Note that each  $|\psi^{(n)}\rangle$  is a full 4-component vector in spinor space; that is, the superscript  $(n)$  indicates which complete vector we're dealing with. Thus  $|\psi^{(n)}\rangle_j$  is the  $j$ th component of the  $n$ th vector.

We can write the 4 PDEs as a matrix eigenvalue equation by moving the terms involving  $m$  to the RHS and factoring out an  $i$  from the terms remaining on the LHS:

$$i \begin{bmatrix} \partial_0 & 0 & \partial_3 & \partial_1 - i\partial_2 \\ 0 & \partial_0 & \partial_1 + i\partial_2 & -\partial_3 \\ -\partial_3 & -\partial_1 + i\partial_2 & -\partial_0 & 0 \\ -\partial_1 - i\partial_2 & \partial_3 & 0 & -\partial_0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = m \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \quad (6)$$

We'll now look at the four solutions  $|\psi^{(n)}\rangle$  and verify that they satisfy 6. First, we have

$$|\psi^{(1)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{bmatrix} e^{-ipx} \equiv u_1 e^{-ipx} \quad (7)$$

where  $u_1$  is defined by this equation as the constant  $\sqrt{\frac{E+m}{2m}}$  multiplied by the 4-d spinor factor. Remember that  $px$  is a 4-vector product:

$$px = p^\mu x_\mu = Et - \mathbf{p} \cdot \mathbf{x} \quad (8)$$

The derivatives in 6 are all with respect to spacetime variables, so act only on  $e^{-ipx}$ ; the spinor components are constants with respect to these derivatives. The first row in 6 is therefore

$$i\sqrt{\frac{E+m}{2m}} e^{-ipx} \left[ -iE + 0 + \frac{p^3}{E+m} (ip^3) + \frac{p^1+ip^2}{E+m} (ip^1+p^2) \right] = \quad (9)$$

$$-\sqrt{\frac{E+m}{2m}} e^{-ipx} \left[ -E + \frac{\mathbf{p}^2}{E+m} \right] = \quad (10)$$

$$-\sqrt{\frac{E+m}{2m}} e^{-ipx} \left[ -E + \frac{E^2 - m^2}{E+m} \right] = \quad (11)$$

$$-\sqrt{\frac{E+m}{2m}} e^{-ipx} \left[ -E + \frac{(E+m)(E-m)}{E+m} \right] = \sqrt{\frac{E+m}{2m}} e^{-ipx} m \quad (12)$$

$$= m\psi_1 \quad (13)$$

Thus the first row of 6 is verified. The other 3 rows can be verified similarly. For row 2:

$$i\sqrt{\frac{E+m}{2m}} e^{-ipx} \left[ 0 + 0 + \frac{p^3}{E+m} (ip^1 - p^2) + \frac{p^1+ip^2}{E+m} (-ip^3) \right] = 0 = m\psi_2 \quad (14)$$

For row 3:

$$i\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[-ip^3+0+\frac{p^3}{E+m}(iE)+0\right]=i\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[\frac{-ip^3(E+m)+ip^3E}{E+m}\right] \quad (15)$$

$$=\sqrt{\frac{E+m}{2m}}e^{-ipx}m\frac{p^3}{E+m} \quad (16)$$

$$=m\psi_3 \quad (17)$$

And for row 4:

$$i\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[(-ip^1+p^2)+0+0+\frac{p^1+ip^2}{E+m}iE\right]= \quad (18)$$

$$\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[(p^1+ip^2)-\frac{p^1+ip^2}{E+m}E\right]= \quad (19)$$

$$\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[\frac{(p^1+ip^2)(E+m)-(p^1+ip^2)E}{E+m}\right]= \quad (20)$$

$$\sqrt{\frac{E+m}{2m}}e^{-ipx}\frac{p^1+ip^2}{E+m}m=m\psi_4 \quad (21)$$

The other 3 solutions are

$$|\psi^{(2)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \end{bmatrix} e^{-ipx} \equiv u_2 e^{-ipx} \quad (22)$$

$$|\psi^{(3)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \\ 1 \\ 0 \end{bmatrix} e^{ipx} \equiv v_2 e^{ipx} \quad (23)$$

$$|\psi^{(4)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \\ 0 \\ 1 \end{bmatrix} e^{ipx} \equiv v_1 e^{ipx} \quad (24)$$

If you really want to, you can verify that these 3 vectors satisfy 6 by grinding through the calculations as above. One point worth noting is that

the constant  $\sqrt{\frac{E+m}{2m}}$  that multiplies all the solutions could be any other constant and still satisfy 6 (since the constant just cancels off both sides). It's chosen to be  $\sqrt{\frac{E+m}{2m}}$  to make later calculations easier.

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