DIRAC EQUATION: INNER PRODUCTS OF SPINORS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problems 4.6 - 4.8.

The four solutions of the Dirac equation in relativistic quantum mechanics are

$$\left|\psi^{(1)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}1\\0\\\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\end{bmatrix}e^{-ipx} \equiv u_1 e^{-ipx} \tag{1}$$

$$\left|\psi^{(2)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}0\\1\\\frac{p^1-ip^2}{E+m}\\-\frac{p^3}{E+m}\end{bmatrix}e^{-ipx} \equiv u_2 e^{-ipx} \tag{2}$$

$$\left|\psi^{(3)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\\1\\0\end{bmatrix}e^{ipx} \equiv v_2 e^{ipx} \tag{3}$$

$$\left|\psi^{(4)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^1-ip^2}{E+m}\\-\frac{p^3}{E+m}\\0\\1\end{bmatrix}e^{ipx} \equiv v_1 e^{ipx} \tag{4}$$

In Klauber's section 4.1.5, he works out the inner product $u_1^{\dagger}(\mathbf{p})\,u_1(\mathbf{p}) = \frac{E}{m}$. The other 3 inner products can be worked out similarly. We get (we are taking all spinors as functions of the same 3-momentum $\mathbf{p}$ in what follows):

$$u_2^{\dagger} = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}0 & 1 & \frac{p^1+ip^2}{E+m} & -\frac{p^3}{E+m}\end{bmatrix} \tag{5}$$

so

$$u_2^\dagger u_2 = \frac{E+m}{2m}\left(1+\frac{\left(p^1+ip^2\right)\left(p^1-ip^2\right)+\left(p^3\right)^2}{\left(E+m\right)^2}\right) \tag{6}$$

$$= \frac{E+m}{2m}\left(1+\frac{\mathbf{p}^2}{\left(E+m\right)^2}\right) \tag{7}$$

$$= \frac{E+m}{2m}\left(\frac{\left(E+m\right)^2+\mathbf{p}^2}{\left(E+m\right)^2}\right) \tag{8}$$

$$= \frac{1}{2m\left(E+m\right)}\left(\left(E+m\right)^2+E^2-m^2\right) \tag{9}$$

$$= \frac{1}{2m\left(E+m\right)}\left(2E^2+2Em\right) \tag{10}$$

$$= \frac{E}{m} \tag{11}$$

For v_1 we note that the first two components of v_1 are the same as the last 2 components of u_2 and vice versa, so we must have

$$v_1^\dagger v_1 = u_2^\dagger u_2 = \frac{E}{m} \tag{12}$$

Similarly, for v_2, the first two components of v_2 are the same as the last 2 components of u_1 and vice versa, so we must have

$$v_2^\dagger v_2 = u_1^\dagger u_1 = \frac{E}{m} \tag{13}$$

The two pairs of spinors are also mutually orthogonal. For example

$$v_1^\dagger v_2 = \frac{E+m}{2m}\left[\frac{\left(p^1+ip^2\right)p^3}{\left(E+m\right)^2}-\frac{\left(p^1+ip^2\right)p^3}{\left(E+m\right)^2}+0+0\right]=0 \tag{14}$$

$$u_1^\dagger u_2 = \frac{E+m}{2m}\left[\frac{\left(p^1-ip^2\right)p^3}{\left(E+m\right)^2}-\frac{\left(p^1-ip^2\right)p^3}{\left(E+m\right)^2}+0+0\right]=0 \tag{15}$$

If one spinor's momentum is the opposite of the other, we get

$$u_1^\dagger\left(\mathbf{p}\right)u_2\left(-\mathbf{p}\right) = \frac{E+m}{2m}\left[\frac{\left(p^1-ip^2\right)\left(-p^3\right)}{\left(E+m\right)^2}-\frac{-\left(p^1-ip^2\right)p^3}{\left(E+m\right)^2}+0+0\right]=0 \tag{16}$$

Also for opposite momenta, we get

$$u_1^\dagger(\mathbf{p})\,v_2(-\mathbf{p}) = \frac{E+m}{2m}\left[\frac{-p^3}{E+m}+0+\frac{p^3}{E+m}+0\right] = 0 \tag{17}$$

$$u_1^\dagger(\mathbf{p})\,v_1(-\mathbf{p}) = \frac{E+m}{2m}\left[\frac{-p^1+ip^2}{E+m}+0+0+\frac{p^1-ip^2}{E+m}\right] = 0 \tag{18}$$

$$u_2^\dagger(\mathbf{p})\,v_2(-\mathbf{p}) = \frac{E+m}{2m}\left[0-\frac{p^1+ip^2}{E+m}+\frac{p^1+ip^2}{E+m}+0\right] = 0 \tag{19}$$

So we get the general result

$$u_r^\dagger(\mathbf{p})\,v_s(-\mathbf{p}) = 0 \tag{20}$$

for all values of r and s.

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