

DIRAC EQUATION: INNER PRODUCTS OF SPINORS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problems 4.6 - 4.8.

The four solutions of the Dirac equation in relativistic quantum mechanics are

$$|\psi^{(1)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{bmatrix} e^{-ipx} \equiv u_1 e^{-ipx} \quad (1)$$

$$|\psi^{(2)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \end{bmatrix} e^{-ipx} \equiv u_2 e^{-ipx} \quad (2)$$

$$|\psi^{(3)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \\ 1 \\ 0 \end{bmatrix} e^{ipx} \equiv v_2 e^{ipx} \quad (3)$$

$$|\psi^{(4)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \\ 0 \\ 1 \end{bmatrix} e^{ipx} \equiv v_1 e^{ipx} \quad (4)$$

In Klauber's section 4.1.5, he works out the inner product $u_1^\dagger(\mathbf{p})u_1(\mathbf{p}) = \frac{E}{m}$. The other 3 inner products can be worked out similarly. We get (we are taking all spinors as functions of the same 3-momentum \mathbf{p} in what follows):

$$u_2^\dagger = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 0 & 1 & \frac{p^1+ip^2}{E+m} & -\frac{p^3}{E+m} \end{bmatrix} \quad (5)$$

so

$$u_2^\dagger u_2 = \frac{E+m}{2m} \left(1 + \frac{(p^1 + ip^2)(p^1 - ip^2) + (p^3)^2}{(E+m)^2} \right) \quad (6)$$

$$= \frac{E+m}{2m} \left(1 + \frac{\mathbf{p}^2}{(E+m)^2} \right) \quad (7)$$

$$= \frac{E+m}{2m} \left(\frac{(E+m)^2 + \mathbf{p}^2}{(E+m)^2} \right) \quad (8)$$

$$= \frac{1}{2m(E+m)} \left((E+m)^2 + E^2 - m^2 \right) \quad (9)$$

$$= \frac{1}{2m(E+m)} (2E^2 + 2Em) \quad (10)$$

$$= \frac{E}{m} \quad (11)$$

For v_1 we note that the first two components of v_1 are the same as the last 2 components of u_2 and vice versa, so we must have

$$v_1^\dagger v_1 = u_2^\dagger u_2 = \frac{E}{m} \quad (12)$$

Similarly, for v_2 , the first two components of v_2 are the same as the last 2 components of u_1 and vice versa, so we must have

$$v_2^\dagger v_2 = u_1^\dagger u_1 = \frac{E}{m} \quad (13)$$

The two pairs of spinors are also mutually orthogonal. For example

$$v_1^\dagger v_2 = \frac{E+m}{2m} \left[\frac{(p^1 + ip^2)p^3}{(E+m)^2} - \frac{(p^1 + ip^2)p^3}{(E+m)^2} + 0 + 0 \right] = 0 \quad (14)$$

$$u_1^\dagger u_2 = \frac{E+m}{2m} \left[\frac{(p^1 - ip^2)p^3}{(E+m)^2} - \frac{(p^1 - ip^2)p^3}{(E+m)^2} + 0 + 0 \right] = 0 \quad (15)$$

If one spinor's momentum is the opposite of the other, we get

$$u_1^\dagger(\mathbf{p}) u_2(-\mathbf{p}) = \frac{E+m}{2m} \left[\frac{(p^1 - ip^2)(-p^3)}{(E+m)^2} - \frac{-(p^1 - ip^2)p^3}{(E+m)^2} + 0 + 0 \right] = 0 \quad (16)$$

Also for opposite momenta, we get

$$u_1^\dagger(\mathbf{p})v_2(-\mathbf{p}) = \frac{E+m}{2m} \left[\frac{-p^3}{E+m} + 0 + \frac{p^3}{E+m} + 0 \right] = 0 \quad (17)$$

$$u_1^\dagger(\mathbf{p})v_1(-\mathbf{p}) = \frac{E+m}{2m} \left[\frac{-p^1 + ip^2}{E+m} + 0 + 0 + \frac{p^1 - ip^2}{E+m} \right] = 0 \quad (18)$$

$$u_2^\dagger(\mathbf{p})v_2(-\mathbf{p}) = \frac{E+m}{2m} \left[0 - \frac{p^1 + ip^2}{E+m} + \frac{p^1 + ip^2}{E+m} + 0 \right] = 0 \quad (19)$$

So we get the general result

$$u_r^\dagger(\mathbf{p})v_s(-\mathbf{p}) = 0 \quad (20)$$

for all values of r and s .

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