

DIRAC EQUATION: ORTHOGONALITY OF SOLUTIONS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.9.

The four solutions of the Dirac equation in relativistic quantum mechanics are

$$\begin{aligned}
 (1) \quad |\psi^{(1)}\rangle &= \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{bmatrix} e^{-ipx} \equiv u_1 e^{-ipx} \\
 (2) \quad |\psi^{(2)}\rangle &= \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \end{bmatrix} e^{-ipx} \equiv u_2 e^{-ipx} \\
 (3) \quad |\psi^{(3)}\rangle &= \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \\ 1 \\ 0 \end{bmatrix} e^{ipx} \equiv v_2 e^{ipx} \\
 (4) \quad |\psi^{(4)}\rangle &= \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \\ 0 \\ 1 \end{bmatrix} e^{ipx} \equiv v_1 e^{ipx}
 \end{aligned}$$

We've explored some of the inner products of the spinor parts of these solutions, so now we can look at the orthogonality properties of the solutions themselves. In his section 4.1.5, Klauber works out a couple of the brackets involving two of the solutions, so we can follow his lead to work out a couple more.

First, we'll look at:

$$(5) \quad \langle \psi^{(1)} | \psi^{(4)} \rangle = u_1^\dagger(\mathbf{p}) v_1(\mathbf{p}) \int e^{ipx} e^{ipx} d^3x$$

$$(6) \quad = u_1^\dagger(\mathbf{p}) v_1(\mathbf{p}) \int e^{2ipx} d^3x$$

$$(7) \quad = 0$$

As with our earlier treatment of the Klein-Gordon equation, we're considering here only systems confined to a finite rectangular box of volume V , so that the allowed momenta \mathbf{p} are discrete values. Because of the boundary conditions that all waves must be zero at the boundaries, the sine and cosine portions of the exponential e^{ipx} must fit an integral or half-integral number of times between the boundaries. This means that e^{2ipx} always fits an integral number of wavelengths between the boundaries, so its integral over each of the 3 spatial components is zero (the positive part of the sine wave always cancels the negative part of the same wave). Therefore we get the zero above. [Note that $u_1^\dagger(\mathbf{p}) v_1(\mathbf{p}) = \frac{1}{m} (p^1 - ip^2) \neq 0$ so we can't use the spinor part to get the zero.]

Secondly, we consider

$$(8) \quad \langle \psi^{(3)} | \psi^{(4)} \rangle = v_2^\dagger(\mathbf{p}) v_1(\mathbf{p}) \int e^{-ipx} e^{ipx} d^3x$$

$$(9) \quad = v_2^\dagger(\mathbf{p}) v_1(\mathbf{p}) \int d^3x$$

$$(10) \quad = v_2^\dagger(\mathbf{p}) v_1(\mathbf{p}) V$$

$$(11) \quad = 0$$

In this case, the integral $\int d^3x$ is just the volume V of the container. However, because of our earlier result, we know that $v_2^\dagger(\mathbf{p}) v_1(\mathbf{p}) = 0$ so here it is the spinor factor that causes the bracket to be zero.

In fact, if we work through all combinations of brackets of unequal solutions, we find that

$$(12) \quad \langle \psi^{(r)} | \psi^{(s)} \rangle = 0 \text{ if } r \neq s$$

However, it is curious that sometimes we get zero because of the orthogonality of the spinors and sometimes because of the integral of the complex exponential over space.

For brackets consisting of equal solutions, we get, using the earlier result that $u_r^\dagger u_r = v_r^\dagger v_r = \frac{E}{m}$ (no sum over r):

$$(13) \quad \langle \psi^{(r)} | \psi^{(r)} \rangle = \frac{E}{m} \int e^{-ipx} e^{ipx} d^3x = \frac{EV}{m}$$

The general result is therefore

$$(14) \quad \langle \psi^{(r)} | \psi^{(s)} \rangle = \frac{EV}{m} \delta_{rs}$$

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