

DIRAC EQUATION: ADJOINT SOLUTIONS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.10.

The four solutions of the Dirac equation in relativistic quantum mechanics are

$$|\psi^{(1)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{bmatrix} e^{-ipx} \equiv u_1 e^{-ipx} \quad (1)$$

$$|\psi^{(2)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \end{bmatrix} e^{-ipx} \equiv u_2 e^{-ipx} \quad (2)$$

$$|\psi^{(3)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \\ 1 \\ 0 \end{bmatrix} e^{ipx} \equiv v_2 e^{ipx} \quad (3)$$

$$|\psi^{(4)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \\ 0 \\ 1 \end{bmatrix} e^{ipx} \equiv v_1 e^{ipx} \quad (4)$$

We've seen that these solutions are mutually orthogonal by taking the inner product of each solution with the complex conjugate transpose of another solution. You might think that we could derive a sort of conjugate transpose version of the Dirac equation by using the 'bra' versions of the solutions, but in fact it seems that it is more usual to define an *adjoint* of each solution by taking the conjugate transpose and then post-multiplying it by the matrix γ^0 . That is, we define the adjoint solutions $\langle \bar{\psi}^{(n)} |$ by

$$\langle \bar{\psi}^{(n)} | \equiv \langle \psi^{(n)} | \gamma^0 \quad (5)$$

with

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (6)$$

We can write out the four adjoints explicitly by taking the conjugate transpose and doing the matrix multiplication. We get

$$\langle \bar{\psi}^{(1)} | = \sqrt{\frac{E+m}{2m}} \left[1 \ 0 \ \frac{p^3}{E+m} \ \frac{p^1-ip^2}{E+m} \right] e^{ipx} \gamma^0 \quad (7)$$

$$= \sqrt{\frac{E+m}{2m}} \left[1 \ 0 \ -\frac{p^3}{E+m} \ -\frac{p^1-ip^2}{E+m} \right] e^{ipx} \quad (8)$$

$$\langle \bar{\psi}^{(2)} | = \sqrt{\frac{E+m}{2m}} \left[0 \ 1 \ -\frac{p^1+ip^2}{E+m} \ \frac{p^3}{E+m} \right] e^{ipx} \quad (9)$$

$$\langle \bar{\psi}^{(3)} | = \sqrt{\frac{E+m}{2m}} \left[\frac{p^3}{E+m} \ \frac{p^1-ip^2}{E+m} \ -1 \ 0 \right] e^{-ipx} \quad (10)$$

$$\langle \bar{\psi}^{(4)} | = \sqrt{\frac{E+m}{2m}} \left[\frac{p^1+ip^2}{E+m} \ -\frac{p^3}{E+m} \ 0 \ -1 \right] e^{-ipx} \quad (11)$$

Note that post-multiplying by γ^0 just changes the sign of the last two elements in $\langle \bar{\psi}^{(n)} |$.

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