DIRAC EQUATION: ADJOINT SOLUTIONS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.10.

The four solutions of the Dirac equation in relativistic quantum mechanics are

$$\left|\psi^{(1)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}1\\0\\\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\end{bmatrix}e^{-ipx}\equiv u_1e^{-ipx} \tag{1}$$

$$\left|\psi^{(2)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}0\\1\\\frac{p^1-ip^2}{E+m}\\-\frac{p^3}{E+m}\end{bmatrix}e^{-ipx}\equiv u_2e^{-ipx} \tag{2}$$

$$\left|\psi^{(3)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\\1\\0\end{bmatrix}e^{ipx}\equiv v_2e^{ipx} \tag{3}$$

$$\left|\psi^{(4)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^1-ip^2}{E+m}\\-\frac{p^3}{E+m}\\0\\1\end{bmatrix}e^{ipx}\equiv v_1e^{ipx} \tag{4}$$

We've seen that these solutions are mutually orthogonal by taking the inner product of each solution with the complex conjugate transpose of another solution. You might think that we could derive a sort of conjugate transpose version of the Dirac equation by using the 'bra' versions of the solutions, but in fact it seems that it is more usual to define an *adjoint* of each solution by taking the conjugate transpose and then post-multiplying it by the matrix γ^0. That is, we define the adjoint solutions $\left\langle\bar{\psi}^{(n)}\right|$ by

$$\left\langle\bar{\psi}^{(n)}\right|\equiv\left\langle\psi^{(n)}\right|\gamma^0 \tag{5}$$

with

$$\gamma^0=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix} \tag{6}$$

We can write out the four adjoints explicitly by taking the conjugate transpose and doing the matrix multiplication. We get

$$\left\langle\bar{\psi}^{(1)}\right| = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}1 & 0 & \frac{p^3}{E+m} & \frac{p^1-ip^2}{E+m}\end{bmatrix}e^{ipx}\gamma^0 \tag{7}$$

$$= \sqrt{\frac{E+m}{2m}}\begin{bmatrix}1 & 0 & -\frac{p^3}{E+m} & -\frac{p^1-ip^2}{E+m}\end{bmatrix}e^{ipx} \tag{8}$$

$$\left\langle\bar{\psi}^{(2)}\right| = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}0 & 1 & -\frac{p^1+ip^2}{E+m} & \frac{p^3}{E+m}\end{bmatrix}e^{ipx} \tag{9}$$

$$\left\langle\bar{\psi}^{(3)}\right| = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^3}{E+m} & \frac{p^1-ip^2}{E+m} & -1 & 0\end{bmatrix}e^{-ipx} \tag{10}$$

$$\left\langle\bar{\psi}^{(4)}\right| = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^1+ip^2}{E+m} & -\frac{p^3}{E+m} & 0 & -1\end{bmatrix}e^{-ipx} \tag{11}$$

Note that post-multiplying by γ^0 just changes the sign of the last two elements in $\left\langle\psi^{(n)}\right|$.

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