ADJOINT DIRAC EQUATION: EXPLICIT SOLUTIONS

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.12.

The adjoint Dirac equation is

$$i\partial_\mu \left\langle \bar{\psi}^{(n)} \right| \gamma^\mu + m \left\langle \bar{\psi}^{(n)} \right| = 0 \tag{1}$$

where there are four solution vectors $n = 1, \ldots, 4$ and the adjoint solutions are given by

$$\left\langle \bar{\psi}^{(n)} \right| = \left\langle \psi^{(n)} \right| \gamma^0 \tag{2}$$

The four adjoint solutions are

$$\left\langle \bar{\psi}^{(1)} \right| = \sqrt{\frac{E+m}{2m}} \left[\begin{array}{cccc} 1 & 0 & -\frac{p^3}{E+m} & -\frac{p^1 - ip^2}{E+m} \end{array} \right] e^{ipx} \tag{3}$$

$$\left\langle \bar{\psi}^{(2)} \right| = \sqrt{\frac{E+m}{2m}} \left[\begin{array}{cccc} 0 & 1 & -\frac{p^1 + ip^2}{E+m} & \frac{p^3}{E+m} \end{array} \right] e^{ipx} \tag{4}$$

$$\left\langle \bar{\psi}^{(3)} \right| = \sqrt{\frac{E+m}{2m}} \left[\begin{array}{cccc} \frac{p^3}{E+m} & \frac{p^1 - ip^2}{E+m} & -1 & 0 \end{array} \right] e^{-ipx} \tag{5}$$

$$\left\langle \bar{\psi}^{(4)} \right| = \sqrt{\frac{E+m}{2m}} \left[\begin{array}{cccc} \frac{p^1 + ip^2}{E+m} & -\frac{p^3}{E+m} & 0 & -1 \end{array} \right] e^{-ipx} \tag{6}$$

Although the derivation of the adjoint equation 1 effectively proves that these four adjoint solutions do in fact solve the adjoint equation, we can show it more directly by substituting the solutions into the equation. Because of the implied sum over μ we'll need the four gamma matrices:

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{7}$$

$$\gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

$$\gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

$$\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{10}$$

The solution vectors all work in much the same way, so we'll look at just $\left\langle \bar{\psi}^{(1)} \right|$ here. The derivatives ∂_μ act only on the e^{ipx} factor, so using

$$px = Ex_0 - \mathbf{p} \cdot \mathbf{x} \tag{11}$$

we get

$$i\partial_0 \left\langle \bar{\psi}^{(1)} \right| \gamma^0 = -E\sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 & 0 & \frac{p^3}{E+m} & \frac{p^1-ip^2}{E+m} \end{bmatrix} e^{ipx} \tag{12}$$

$$i\partial_1 \left\langle \bar{\psi}^{(1)} \right| \gamma^1 = p^1\sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^1-ip^2}{E+m} & \frac{p^3}{E+m} & 0 & 1 \end{bmatrix} e^{ipx} \tag{13}$$

$$i\partial_2 \left\langle \bar{\psi}^{(1)} \right| \gamma^2 = p^2\sqrt{\frac{E+m}{2m}} \begin{bmatrix} i\frac{p^1-ip^2}{E+m} & -i\frac{p^3}{E+m} & 0 & -i \end{bmatrix} e^{ipx} \tag{14}$$

$$i\partial_3 \left\langle \bar{\psi}^{(1)} \right| \gamma^3 = p^3\sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^3}{E+m} & -\frac{p^1-ip^2}{E+m} & 1 & 0 \end{bmatrix} e^{ipx} \tag{15}$$

Adding together the first component from each of these equations, we get

$$\sqrt{\frac{E+m}{2m}}e^{ipx}\left[-E+\frac{\left(p^1\right)^2-ip^1p^2+ip^1p^2+\left(p^2\right)^2+\left(p^3\right)^2}{E+m}\right]=\sqrt{\frac{E+m}{2m}}e^{ipx}\left(-E+\frac{\mathbf{p}^2}{E+m}\right) \tag{16}$$

$$=\sqrt{\frac{E+m}{2m}}e^{ipx}\left(-E+\frac{E^2-m^2}{E+m}\right) \tag{17}$$

$$=-\sqrt{\frac{E+m}{2m}}e^{ipx}m \tag{18}$$

For the second component, we get

$$\sqrt{\frac{E+m}{2m}}e^{ipx}\left[\frac{0+p^1p^3-ip^2p^3-p^1p^3+ip^2p^3}{E+m}\right]=0 \tag{19}$$

For the third component:

$$\sqrt{\frac{E+m}{2m}}e^{ipx}\left[\frac{-Ep^3+0+0}{E+m}+p^3\right]=\sqrt{\frac{E+m}{2m}}e^{ipx}\left[\frac{-Ep^3+p^3\left(E+m\right)}{E+m}\right] \tag{20}$$

$$=m\sqrt{\frac{E+m}{2m}}e^{ipx}\frac{p^3}{E+m} \tag{21}$$

And for the fourth component

$$\sqrt{\frac{E+m}{2m}}e^{ipx}\left[\frac{-E\left(p^1-ip^2\right)}{E+m}+p^1-ip^2+0\right]=\sqrt{\frac{E+m}{2m}}e^{ipx}\left[\frac{\left(-E+E+m\right)\left(p^1-ip^2\right)}{E+m}\right] \tag{22}$$

$$=m\sqrt{\frac{E+m}{2m}}e^{ipx}\frac{p^1-ip^2}{E+m} \tag{23}$$

Comparing these results with the original adjoint solution $\left\langle\bar{\psi}^{(1)}\right|$ we see that

$$i\partial_\mu\left\langle\bar{\psi}^{(1)}\right|\gamma^\mu=-m\left\langle\bar{\psi}^{(1)}\right| \tag{24}$$

so $\left\langle\bar{\psi}^{(1)}\right|$ does indeed satisfy the adjoint Dirac equation. The other 3 solutions work similarly.