DIRAC EQUATION: CONSERVED PROBABILITY CURRENT

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.13.

The Dirac equation is

$$(i\gamma^{\mu}\partial_{\mu}-m)|\psi\rangle=0 \tag{1}$$

and the adjoint Dirac equation is

$$i\partial_{\mu}\left\langle \bar{\psi}\right|\gamma^{\mu}+m\left\langle \bar{\psi}\right|=0 \tag{2}$$

where there are four solution vectors $n=1,\ldots,4$ and the adjoint solutions are given by

$$\left\langle \bar{\psi}\right|=\langle\psi|\gamma^{0} \tag{3}$$

We'd like to find a conserved quantity analogous to that for the Klein-Gordon equation. We can do this as follows. First we multiply 1 on the left by the adjoint solutions:

$$i\left\langle \bar{\psi}\left|\gamma^{\mu}\partial_{\mu}\right|\psi\right\rangle =m\left\langle \bar{\psi}\left|\psi\right.\right\rangle \tag{4}$$

Then, multiply 2 on the right by the original solutions:

$$i\left(\partial_{\mu}\left\langle \bar{\psi}\right|\right)\gamma^{\mu}\left|\psi\right\rangle =-m\left\langle \bar{\psi}\left|\psi\right.\right\rangle \tag{5}$$

We've kept the bra in parentheses since the derivative applies only to it and not the ket portion. Adding these two equations gives

$$\begin{aligned} i\left\langle \bar{\psi}\left|\gamma^{\mu}\partial_{\mu}\right|\psi\right\rangle +i\left(\partial_{\mu}\left\langle \bar{\psi}\right|\right)\gamma^{\mu}\left|\psi\right\rangle & = 0 & (6)\\ i\partial_{\mu}\left\langle \bar{\psi}\left|\gamma^{\mu}\right|\psi\right\rangle & = 0 & (7)\end{aligned}$$

where we've used the product rule to combine the two derivatives, so that the ∂_{μ} in the last line *does* apply to the full bracket. Note also that in the bracket in the last line, there is no integration over space, since all we've done is multiply the adjoint solution into the regular solution.

By the way, you might think that this result is trivial, since spacetime enters into $|\psi\rangle$ only in the form $e^{\pm ipx}$ and therefore into $\left\langle \bar{\psi}\right|$ in the form $e^{\mp ipx}$, so it would seem that the bracket $\left\langle \bar{\psi}\left|\gamma^{\mu}\right|\psi\right\rangle$ has no dependence on x so obviously its derivative must be zero. However, the $|\psi\rangle$ here can refer to a sum of states with different momenta **p**, so the result isn't quite as trivial as it looks.

We can therefore define a conserved current j^{μ} as

$$j^{\mu}\equiv\left\langle \bar{\psi}\left|\gamma^{\mu}\right|\psi\right\rangle \tag{8}$$

so that

$$\partial_{\mu}j^{\mu}=0 \tag{9}$$

Again, remember that there is no integration over space in the definition of j^{μ}.

In Klauber's equations 4-36 and 4-37, he shows that if we consider a single particle in the state

$$|\psi\rangle=\sum_{r,\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}C_{r}\left(\mathbf{p}\right)u_{r}e^{-ipx} \tag{10}$$

(remember that r ranges over the four solutions and **p** over all possible discrete momenta) and integrate $\rho\equiv j^{0}$ over space, we get the condition

$$\sum_{r,\mathbf{p}}\left|C_{r}\left(\mathbf{p}\right)\right|^{2}=1 \tag{11}$$

Thus we can interpret $\left|C_{r}\left(\mathbf{p}\right)\right|^{2}$ as the probability of finding the particle in state r with momentum **p**.

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