DIRAC EQUATION: POSITIVE PROBABILITIES AND NEGATIVE ENERGIES

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.14.

One problem with the Klein-Gordon equation in relativistic quantum mechanics is that the probability of finding the system in some state can be positive (for particles) or negative (for antiparticles). The Dirac equation solves this problem, as Klauber shows in his equations 4-36 through 4-38. We'll summarize these results here.

The solutions to the Dirac equation are

$$\left|\psi^{(1)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}1\\0\\\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\end{bmatrix}e^{-ipx}\equiv u_1e^{-ipx} \tag{1}$$

$$\left|\psi^{(2)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}0\\1\\\frac{p^1-ip^2}{E+m}\\-\frac{p^3}{E+m}\end{bmatrix}e^{-ipx}\equiv u_2e^{-ipx} \tag{2}$$

$$\left|\psi^{(3)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\\1\\0\end{bmatrix}e^{ipx}\equiv v_2e^{ipx} \tag{3}$$

$$\left|\psi^{(4)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^1-ip^2}{E+m}\\-\frac{p^3}{E+m}\\0\\1\end{bmatrix}e^{ipx}\equiv v_1e^{ipx} \tag{4}$$

Since the Dirac equation is linear, any linear combination of these solutions is also a solution, so the most general solution is

$$|\psi\rangle=\sum_{\mathbf{p}}\sum_{r=1}^{2}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[C_r(\mathbf{p})\,u_r(\mathbf{p})\,e^{-ipx}+D_r^{\dagger}(\mathbf{p})\,v_r(\mathbf{p})\,e^{ipx}\right] \tag{5}$$

where C_r and D_r are constant (independent of x) coefficients.

The probability density ρ for the Dirac equation is

$$\rho=\left\langle\bar{\psi}\left|\gamma^0\right|\psi\right\rangle \tag{6}$$

where $\langle\bar{\psi}|$ is an adjoint solution and despite the bracket notation, there is no integration over space involved in the definition of ρ. The *total* probability of the system being in some state can be found by integrating ρ over space. This involves using the orthogonality of the solutions and inner products of the spinors (details in Klauber as mentioned above). The results for a state that consists entirely of C_r terms (all D_r are zero) is that

$$\int\rho d^3x=\sum_{r,\mathbf{p}}|C_r(\mathbf{p})|^2=1 \tag{7}$$

where we have imposed the "= 1" since the total probability must be one.

For a state consisting entirely of D_r terms (all C_r are zero), we get the result

$$\int\rho d^3x=\sum_{r,\mathbf{p}}|D_r(\mathbf{p})|^2=1 \tag{8}$$

The crucial difference between Klein-Gordon and Dirac is that with Dirac, *both* probabilities are positive.

The Dirac equation does, however, still give both positive and negative energies for a free particle. The Hamiltonian can be written using the operator $i\frac{\partial}{\partial t}$ so applying this to 1 or 2 above we get

$$i\frac{\partial}{\partial t}\left(u_{1,2}e^{-ipx}\right)=p^0u_{1,2}e^{-ipx}=+E_{\mathbf{p}}u_{1,2}e^{-ipx} \tag{9}$$

And applying to 3 or 4 we get

$$i\frac{\partial}{\partial t}\left(v_{1,2}e^{ipx}\right)=-p^0v_{1,2}e^{ipx}=-E_{\mathbf{p}}v_{1,2}e^{ipx} \tag{10}$$

Thus those terms in 5 with C_r coefficients have positive energy and terms with D_r coefficients have negative energy.

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