EIGENSPINORS OF THE PAULI SPIN MATRICES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.15.

In non-relativistic quantum mechanics, the spin $\frac{1}{2}$ operators are given in terms of the Pauli matrices as

$$S_i = \frac{\hbar}{2}\sigma_i \tag{1}$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{3}$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4}$$

We've seen that the spinor states $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenstates of the S_z operator, and that these spinors form a basis for the 2-d spinor space. We can find the eigenstates of S_x and S_y in the usual way from matrix algebra. For σ_x, the eigenvalues are

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \tag{5}$$

$$\lambda = \pm 1 \tag{6}$$

For $\lambda = 1$, the eigenvector equation is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{7}$$

which gives

$$a = b \tag{8}$$

To normalize the eigenstate, we can choose $a = b = \frac{1}{\sqrt{2}}$, so that the state is $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda=-1$, we get

$$\left[\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right]\left[\begin{array}{c}a\\ b\end{array}\right]=-\left[\begin{array}{c}a\\ b\end{array}\right] \tag{9}$$

so the normalized eigenstate is $\frac{1}{\sqrt{2}}\left[\begin{array}{c}1\\ -1\end{array}\right]$.

For σ_y, we get

$$\begin{aligned}\left|\begin{array}{cc}-\lambda & -i\\ i & -\lambda\end{array}\right| &= \lambda^2-1=0 \quad (10)\\ \lambda &= \pm1 \quad (11)\end{aligned}$$

so the eigenvalues are the same as for σ_x and σ_z. The eigenvector equations are

$$\left[\begin{array}{cc}0 & -i\\ i & 0\end{array}\right]\left[\begin{array}{c}a\\ b\end{array}\right]=\pm\left[\begin{array}{c}a\\ b\end{array}\right] \tag{12}$$

from which we get

$$a=\mp ib \tag{13}$$

Thus the two normalized eigenstates are, for $\lambda=+1,\ -1$ respectively:

$$\frac{1}{\sqrt{2}}\left[\begin{array}{c}i\\ -1\end{array}\right],\ \frac{1}{\sqrt{2}}\left[\begin{array}{c}i\\ 1\end{array}\right] \tag{14}$$

This can be written in terms of the σ_z eigenstates as

$$\frac{1}{\sqrt{2}}\left[\begin{array}{c}i\\ -1\end{array}\right]=\frac{i}{\sqrt{2}}\left[\begin{array}{c}1\\ 0\end{array}\right]+\frac{1}{\sqrt{2}}\left[\begin{array}{c}0\\ -1\end{array}\right] \tag{15}$$

Since the coefficients of the two σ_z eigenstates are equal in magnitude, this means that if σ_z is measured for a particle in an σ_y eigenstate, it is equally likely to be spin up or spin down. The same applies to a particle in a σ_x eigenstate.

PINGBACKS

Pingback: Dirac equation: spin of a particle at rest
Pingback: Dirac equation: non-relativistic limit