

## EIGENSPINORS OF THE PAULI SPIN MATRICES

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.15.

In non-relativistic quantum mechanics, the spin  $\frac{1}{2}$  operators are given in terms of the Pauli matrices as

$$S_i = \frac{\hbar}{2}\sigma_i \quad (1)$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (2)$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (3)$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (4)$$

We've seen that the spinor states  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  are eigenstates of the  $S_z$  operator, and that these spinors form a basis for the 2-d spinor space. We can find the eigenstates of  $S_x$  and  $S_y$  in the usual way from matrix algebra. For  $\sigma_x$ , the eigenvalues are

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \quad (5)$$

$$\lambda = \pm 1 \quad (6)$$

For  $\lambda = 1$ , the eigenvector equation is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad (7)$$

which gives

$$a = b \quad (8)$$

To normalize the eigenstate, we can choose  $a = b = \frac{1}{\sqrt{2}}$ , so that the state is  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

For  $\lambda = -1$ , we get

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = - \begin{bmatrix} a \\ b \end{bmatrix} \quad (9)$$

so the normalized eigenstate is  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

For  $\sigma_y$ , we get

$$\begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \quad (10)$$

$$\lambda = \pm 1 \quad (11)$$

so the eigenvalues are the same as for  $\sigma_x$  and  $\sigma_z$ . The eigenvector equations are

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \pm \begin{bmatrix} a \\ b \end{bmatrix} \quad (12)$$

from which we get

$$a = \mp ib \quad (13)$$

Thus the two normalized eigenstates are, for  $\lambda = +1, -1$  respectively:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} i \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix} \quad (14)$$

This can be written in terms of the  $\sigma_z$  eigenstates as

$$\frac{1}{\sqrt{2}} \begin{bmatrix} i \\ -1 \end{bmatrix} = \frac{i}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad (15)$$

Since the coefficients of the two  $\sigma_z$  eigenstates are equal in magnitude, this means that if  $\sigma_z$  is measured for a particle in an  $\sigma_y$  eigenstate, it is equally likely to be spin up or spin down. The same applies to a particle in a  $\sigma_x$  eigenstate.

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