DIRAC EQUATION: SPIN OF A PARTICLE AT REST

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.16.

The solutions to the Dirac equation consist of a 4-element column spinor and a spacetime component:

$$\left|\psi^{(1)}\right\rangle=\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c}1\\0\\\frac{p^{3}}{E+m}\\\frac{p^{1}+ip^{2}}{E+m}\end{array}\right]e^{-ipx}\equiv u_{1}e^{-ipx} \tag{1}$$

$$\left|\psi^{(2)}\right\rangle=\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c}0\\1\\\frac{p^{1}-ip^{2}}{E+m}\\-\frac{p^{3}}{E+m}\end{array}\right]e^{-ipx}\equiv u_{2}e^{-ipx} \tag{2}$$

$$\left|\psi^{(3)}\right\rangle=\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c}\frac{p^{3}}{E+m}\\\frac{p^{1}+ip^{2}}{E+m}\\1\\0\end{array}\right]e^{ipx}\equiv v_{2}e^{ipx} \tag{3}$$

$$\left|\psi^{(4)}\right\rangle=\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c}\frac{p^{1}-ip^{2}}{E+m}\\-\frac{p^{3}}{E+m}\\0\\1\end{array}\right]e^{ipx}\equiv v_{1}e^{ipx} \tag{4}$$

The last time we found quantum mechanical solutions that contained column vectors, we introduced these solutions to account for spin in non-relativistic quantum mechanics. With the Dirac equation, spin finds its way into the solutions by being a consequence of the nature of the solutions, rather than by being imposed.

An experimental fact about spin is that the intrinsic spin of a particle is not affected by how fast it is moving. That is, an electron always has spin $\frac{1}{2}$ whether it is observed at rest or moving close to the speed of light. However, due to length contraction, the direction of an angular momentum vector for a spinning object does vary with velocity relative to the observer. This is explained more fully in Klauber's Box 4-2, which looks at the effect of length contraction on a classical (non-quantum) rotating disk. In summary, suppose the angular momentum vector $\mathbf{L}$ lies in the $x-z$ plane at some angle θ to the x axis, so that the plane of the disk, being perpendicular to $\mathbf{L}$, makes the same angle θ with the z axis. Now suppose the disk moves along the x axis at some speed v. As v becomes relativistic, the angle θ diminishes, since the size component of the disk in the x direction is contracted, while the component in the z direction remains unchanged. When $v \to c$, the disk's x component tends to zero, so that $\mathbf{L}$ lies along the x axis and the disk spins in the yz plane.

Because both E and $\mathbf{p}$ depend ultimately on v in the solutions above, the Dirac solutions actually already contain this relativistic effect within the spinor components. To see this, we need the spin operators in the Dirac theory, in analogy to the Pauli matrices for non-relativistic spin $\frac{1}{2}$. For now, we will take these operators to be god-given. They are

$$\Sigma_i = \frac{1}{2}\left[\begin{array}{cc} \sigma_i & 0 \\ 0 & \sigma_i \end{array}\right]; \; i = x,y,z \tag{5}$$

(using natural units where $\hbar = 1$) and where the Pauli matrices are

$$\sigma_x = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \tag{6}$$

$$\sigma_y = \left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right] \tag{7}$$

$$\sigma_z = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right] \tag{8}$$

and the 0 components in 5 are 2×2 zero matrices.

As a simple example of how the Σ_i operators work, consider the special case of a particle at rest, so that $\mathbf{p} = 0$ and $E = m$. Then the four solutions at the top reduce to

$$\left|\psi^{(1)}\right\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-imt} \tag{9}$$

$$\left|\psi^{(2)}\right\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-imt} \tag{10}$$

$$\left|\psi^{(3)}\right\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{imt} \tag{11}$$

$$\left|\psi^{(4)}\right\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{imt} \tag{12}$$

These four solutions are all eigenstates of Σ_z as we can see by writing out Σ_z explicitly

$$\Sigma_z = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{13}$$

Then we find

$$\Sigma_z\left|\psi^{(1)}\right\rangle = \frac{1}{2}\left|\psi^{(1)}\right\rangle \tag{14}$$

$$\Sigma_z\left|\psi^{(2)}\right\rangle = -\frac{1}{2}\left|\psi^{(2)}\right\rangle \tag{15}$$

$$\Sigma_z\left|\psi^{(3)}\right\rangle = \frac{1}{2}\left|\psi^{(3)}\right\rangle \tag{16}$$

$$\Sigma_z\left|\psi^{(4)}\right\rangle = -\frac{1}{2}\left|\psi^{(4)}\right\rangle \tag{17}$$

Thus $\left|\psi^{(1)}\right\rangle$ and $\left|\psi^{(3)}\right\rangle$ are eigenstates of a particle with z spin component $\frac{1}{2}$, and $\left|\psi^{(2)}\right\rangle$ and $\left|\psi^{(4)}\right\rangle$ represent a particle with z spin component $-\frac{1}{2}$. As you might expect, the first two solutions are for particles and the last two are for antiparticles, although we haven't actually demonstrated this yet.

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