

DIRAC EQUATION: SPIN OF A MOVING PARTICLE

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.17.

The solutions to the Dirac equation consist of a 4-element column spinor and a spacetime component:

$$|\psi^{(1)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{bmatrix} e^{-ipx} \equiv u_1 e^{-ipx} \quad (1)$$

$$|\psi^{(2)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \end{bmatrix} e^{-ipx} \equiv u_2 e^{-ipx} \quad (2)$$

$$|\psi^{(3)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \\ 1 \\ 0 \end{bmatrix} e^{ipx} \equiv v_2 e^{ipx} \quad (3)$$

$$|\psi^{(4)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \\ 0 \\ 1 \end{bmatrix} e^{ipx} \equiv v_1 e^{ipx} \quad (4)$$

For a particle at rest ($\mathbf{p} = 0$, $E = m$), all four solutions are eigenstates of the spin operator Σ_z , with eigenvalues (spins) of $\pm\frac{1}{2}$. Here, we have a look at what happens if the particle is moving.

First, suppose the particle is moving in the x direction, so that $p^1 \neq 0$ and $p^2 = p^3 = 0$. In this case, the spinor components (which we'll call $s^{(n)}$ for $n = 1, \dots, 4$) of the solutions above become

$$s^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{p^1}{E+m} \end{bmatrix} \quad (5)$$

$$s^{(2)} = \begin{bmatrix} 0 \\ 1 \\ \frac{p^1}{E+m} \\ 0 \end{bmatrix} \quad (6)$$

$$s^{(3)} = \begin{bmatrix} 0 \\ \frac{p^1}{E+m} \\ 1 \\ 0 \end{bmatrix} \quad (7)$$

$$s^{(4)} = \begin{bmatrix} \frac{p^1}{E+m} \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (8)$$

The z spin operator is

$$\Sigma_z = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (9)$$

Multiplying Σ_z into the four $s^{(n)}$ spinors, we get

$$\Sigma_z s^{(1)} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{p^1}{E+m} \end{bmatrix} \quad (10)$$

$$\Sigma_z s^{(2)} = \frac{1}{2} \begin{bmatrix} 0 \\ -1 \\ \frac{p^1}{E+m} \\ 0 \end{bmatrix} \quad (11)$$

$$\Sigma_z s^{(3)} = \frac{1}{2} \begin{bmatrix} 0 \\ -\frac{p^1}{E+m} \\ 1 \\ 0 \end{bmatrix} \quad (12)$$

$$\Sigma_z s^{(4)} = \frac{1}{2} \begin{bmatrix} \frac{p^1}{E+m} \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad (13)$$

In each case, one of the non-zero components of $s^{(n)}$ has its sign changed, while the other non-zero component remains the same. Thus none of the $s^{(n)}$ spinors is an eigenstate of Σ_z .

Now suppose that $p^1 = p^2 = 0$ and $p^3 \neq 0$. The spinors are now

$$s^{(1)} = \begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ 0 \end{bmatrix} \quad (14)$$

$$s^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\frac{p^3}{E+m} \end{bmatrix} \quad (15)$$

$$s^{(3)} = \begin{bmatrix} \frac{p^3}{E+m} \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (16)$$

$$s^{(4)} = \begin{bmatrix} 0 \\ -\frac{p^3}{E+m} \\ 0 \\ 1 \end{bmatrix} \quad (17)$$

Multiplying by Σ_z we get

$$\Sigma_z s^{(1)} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ 0 \end{bmatrix} = \frac{1}{2} s^{(1)} \quad (18)$$

$$\Sigma_z s^{(2)} = \frac{1}{2} \begin{bmatrix} 0 \\ -1 \\ 0 \\ \frac{p^3}{E+m} \end{bmatrix} = -\frac{1}{2} s^{(2)} \quad (19)$$

$$\Sigma_z s^{(3)} = \frac{1}{2} \begin{bmatrix} \frac{p^3}{E+m} \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} s^{(3)} \quad (20)$$

$$\Sigma_z s^{(4)} = \frac{1}{2} \begin{bmatrix} 0 \\ \frac{p^3}{E+m} \\ 0 \\ -1 \end{bmatrix} = -\frac{1}{2} s^{(4)} \quad (21)$$

Thus the $s^{(n)}$ spinors in this case *are* eigenstates of Σ_z .

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