DIRAC EQUATION: SPIN OF A MOVING PARTICLE

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.17.

The solutions to the Dirac equation consist of a 4-element column spinor and a spacetime component:

$$\left|\psi^{(1)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}1\\0\\\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\end{bmatrix}e^{-ipx} \equiv u_1 e^{-ipx} \tag{1}$$

$$\left|\psi^{(2)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}0\\1\\\frac{p^1-ip^2}{E+m}\\-\frac{p^3}{E+m}\end{bmatrix}e^{-ipx} \equiv u_2 e^{-ipx} \tag{2}$$

$$\left|\psi^{(3)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\\1\\0\end{bmatrix}e^{ipx} \equiv v_2 e^{ipx} \tag{3}$$

$$\left|\psi^{(4)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^1-ip^2}{E+m}\\-\frac{p^3}{E+m}\\0\\1\end{bmatrix}e^{ipx} \equiv v_1 e^{ipx} \tag{4}$$

For a particle at rest ($\mathbf{p}=0,\ E=m$), all four solutions are eigenstates of the spin operator Σ_z, with eigenvalues (spins) of $\pm\frac{1}{2}$. Here, we have a look at what happens if the particle is moving.

First, suppose the particle is moving in the x direction, so that $p^1\neq 0$ and $p^2=p^3=0$. In this case, the spinor components (which we'll call $s^{(n)}$ for $n=1,\ldots,4$) of the solutions above become

$$s^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{p^1}{E+m} \end{bmatrix} \tag{5}$$

$$s^{(2)} = \begin{bmatrix} 0 \\ 1 \\ \frac{p^1}{E+m} \\ 0 \end{bmatrix} \tag{6}$$

$$s^{(3)} = \begin{bmatrix} 0 \\ \frac{p^1}{E+m} \\ 1 \\ 0 \end{bmatrix} \tag{7}$$

$$s^{(4)} = \begin{bmatrix} \frac{p^1}{E+m} \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{8}$$

The z spin operator is

$$\Sigma_z = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{9}$$

Multiplying Σ_z into the four $s^{(n)}$ spinors, we get

$$\Sigma_z s^{(1)} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{p^1}{E+m} \end{bmatrix} \tag{10}$$

$$\Sigma_z s^{(2)} = \frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ \frac{p^1}{E+m} \\ 0 \end{bmatrix} \tag{11}$$

$$\Sigma_z s^{(3)} = \frac{1}{2}\begin{bmatrix} 0 \\ -\frac{p^1}{E+m} \\ 1 \\ 0 \end{bmatrix} \tag{12}$$

$$\Sigma_z s^{(4)} = \frac{1}{2}\begin{bmatrix} \frac{p^1}{E+m} \\ 0 \\ 0 \\ -1 \end{bmatrix} \tag{13}$$

In each case, one of the non-zero components of $s^{(n)}$ has its sign changed, while the other non-zero component remains the same. Thus none of the $s^{(n)}$ spinors is an eigenstate of Σ_z.

Now suppose that $p^1 = p^2 = 0$ and $p^3 \neq 0$. The spinors are now

$$s^{(1)} = \begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ 0 \end{bmatrix} \tag{14}$$

$$s^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\frac{p^3}{E+m} \end{bmatrix} \tag{15}$$

$$s^{(3)} = \begin{bmatrix} \frac{p^3}{E+m} \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{16}$$

$$s^{(4)} = \begin{bmatrix} 0 \\ -\frac{p^3}{E+m} \\ 0 \\ 1 \end{bmatrix} \tag{17}$$

Multiplying by Σ_z we get

$$\Sigma_z s^{(1)} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ 0 \end{bmatrix} = \frac{1}{2}s^{(1)} \tag{18}$$

$$\Sigma_z s^{(2)} = \frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ 0 \\ \frac{p^3}{E+m} \end{bmatrix} = -\frac{1}{2}s^{(2)} \tag{19}$$

$$\Sigma_z s^{(3)} = \frac{1}{2}\begin{bmatrix} \frac{p^3}{E+m} \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}s^{(3)} \tag{20}$$

$$\Sigma_z s^{(4)} = \frac{1}{2}\begin{bmatrix} 0 \\ \frac{p^3}{E+m} \\ 0 \\ -1 \end{bmatrix} = -\frac{1}{2}s^{(4)} \tag{21}$$

Thus the $s^{(n)}$ spinors in this case *are* eigenstates of Σ_z.

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