

DIRAC EQUATION: SPINORS NEAR THE SPEED OF LIGHT

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.18.

The solutions to the Dirac equation consist of a 4-element column spinor and a spacetime component:

$$|\psi^{(1)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{bmatrix} e^{-ipx} \equiv u_1 e^{-ipx} \quad (1)$$

$$|\psi^{(2)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \end{bmatrix} e^{-ipx} \equiv u_2 e^{-ipx} \quad (2)$$

$$|\psi^{(3)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \\ 1 \\ 0 \end{bmatrix} e^{ipx} \equiv v_2 e^{ipx} \quad (3)$$

$$|\psi^{(4)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \\ 0 \\ 1 \end{bmatrix} e^{ipx} \equiv v_1 e^{ipx} \quad (4)$$

For a particle at rest ($\mathbf{p} = 0$, $E = m$), or for a particle moving in the z direction, all four solutions are eigenstates of the spin operator Σ_z , with eigenvalues (spins) of $\pm\frac{1}{2}$. If the particle is moving in the x or y direction, the individual spinors above aren't eigenstates of any of the spin operators. As the speed of the particle approaches c , however, we can get some eigenstates of Σ_x and Σ_y .

First, suppose the particle is moving in the x direction at a speed approaching c , with any motion in the y and z directions much smaller by comparison. In this case, $E \rightarrow p^1 \rightarrow \infty$ and the spinor components (which we'll call $s^{(n)}$ for $n = 1, \dots, 4$) of the solutions above become

$$s^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (5)$$

$$s^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad (6)$$

$$s^{(3)} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad (7)$$

$$s^{(4)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (8)$$

The x spin operator is

$$\Sigma_x = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (9)$$

Multiplying Σ_x into $s^{(1)} + s^{(2)}$, we get

$$\Sigma_x (s^{(1)} + s^{(2)}) = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad (10)$$

$$= \frac{1}{2} (s^{(1)} + s^{(2)}) \quad (11)$$

Similarly

$$\Sigma_x (s^{(3)} + s^{(4)}) = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad (12)$$

$$= \frac{1}{2} (s^{(3)} + s^{(4)}) \quad (13)$$

Thus the sums $u_1 + u_2$ and $v_1 + v_2$ are both eigenstates of Σ_x with eigenvalue $\frac{1}{2}$.

Now suppose the particle is moving in the y direction with $v^y \rightarrow 1$ so that $E \rightarrow p^2 \rightarrow \infty$. The four spinors now become

$$s^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ i \end{bmatrix} \quad (14)$$

$$s^{(2)} = \begin{bmatrix} 0 \\ 1 \\ -i \\ 0 \end{bmatrix} \quad (15)$$

$$s^{(3)} = \begin{bmatrix} 0 \\ i \\ 1 \\ 0 \end{bmatrix} \quad (16)$$

$$s^{(4)} = \begin{bmatrix} -i \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (17)$$

The y spin operator is

$$\Sigma_y = \frac{1}{2} \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \quad (18)$$

In this case

$$\Sigma_y \left(s^{(1)} + s^{(3)} \right) = \frac{1}{2} \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ 1 \\ i \end{bmatrix} \quad (19)$$

$$= \frac{1}{2} \begin{bmatrix} 1 \\ i \\ 1 \\ i \end{bmatrix} \quad (20)$$

$$\Sigma_y \left(s^{(2)} + s^{(4)} \right) = \frac{1}{2} \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \begin{bmatrix} -i \\ 1 \\ -i \\ 1 \end{bmatrix} \quad (21)$$

$$= \frac{1}{2} \begin{bmatrix} -i \\ 1 \\ -i \\ 1 \end{bmatrix} \quad (22)$$

Thus the sums $u_1 + v_2$ and $v_1 + u_2$ are both eigenstates of Σ_y with eigenvalue $\frac{1}{2}$. [As the u_j spinors are supposed to represent particles and the v_j antiparticles, I'm not sure what a mixture of the two is supposed to represent.]