DIRAC EQUATION: SPINORS NEAR THE SPEED OF LIGHT

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.18.

The solutions to the Dirac equation consist of a 4-element column spinor and a spacetime component:

$$\left|\psi^{(1)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{bmatrix} e^{-ipx} \equiv u_1 e^{-ipx} \tag{1}$$

$$\left|\psi^{(2)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \end{bmatrix} e^{-ipx} \equiv u_2 e^{-ipx} \tag{2}$$

$$\left|\psi^{(3)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \\ 1 \\ 0 \end{bmatrix} e^{ipx} \equiv v_2 e^{ipx} \tag{3}$$

$$\left|\psi^{(4)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \\ 0 \\ 1 \end{bmatrix} e^{ipx} \equiv v_1 e^{ipx} \tag{4}$$

For a particle at rest ($\mathbf{p}=0$, $E=m$), or for a particle moving in the z direction, all four solutions are eigenstates of the spin operator Σ_z, with eigenvalues (spins) of $\pm\frac{1}{2}$. If the particle is moving in the x or y direction, the individual spinors above aren't eigenstates of any of the spin operators. As the speed of the particle approaches c, however, we can get some eigenstates of Σ_x and Σ_y.

First, suppose the particle is moving in the x direction at a speed approaching c, with any motion in the y and z directions much smaller by comparison. In this case, $E\to p^1\to\infty$ and the spinor components (which we'll call $s^{(n)}$ for $n=1,\ldots,4$) of the solutions above become

$$s^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

$$s^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \tag{6}$$

$$s^{(3)} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \tag{7}$$

$$s^{(4)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{8}$$

The x spin operator is

$$\Sigma_x = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{9}$$

Multiplying Σ_x into $s^{(1)} + s^{(2)}$, we get

$$\Sigma_x \left(s^{(1)} + s^{(2)}\right) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{10}$$

$$= \frac{1}{2}\left(s^{(1)} + s^{(2)}\right) \tag{11}$$

Similarly

$$\Sigma_x\left(s^{(3)}+s^{(4)}\right) = \frac{1}{2}\left[\begin{array}{c}1\\1\\1\\1\end{array}\right] \tag{12}$$

$$= \frac{1}{2}\left(s^{(3)}+s^{(4)}\right) \tag{13}$$

Thus the sums u_1+u_2 and v_1+v_2 are both eigenstates of Σ_x with eigenvalue $\frac{1}{2}$.

Now suppose the particle is moving in the y direction with $v^y\to 1$ so that $E\to p^2\to\infty$. The four spinors now become

$$s^{(1)} = \left[\begin{array}{c}1\\0\\0\\i\end{array}\right] \tag{14}$$

$$s^{(2)} = \left[\begin{array}{c}0\\1\\-i\\0\end{array}\right] \tag{15}$$

$$s^{(3)} = \left[\begin{array}{c}0\\i\\1\\0\end{array}\right] \tag{16}$$

$$s^{(4)} = \left[\begin{array}{c}-i\\0\\0\\1\end{array}\right] \tag{17}$$

The y spin operator is

$$\Sigma_y = \frac{1}{2}\left[\begin{array}{cccc}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{array}\right] \tag{18}$$

In this case

$$\Sigma_y\left(s^{(1)}+s^{(3)}\right)=\frac{1}{2}\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{bmatrix}\begin{bmatrix}1\\i\\1\\i\end{bmatrix} \tag{19}$$

$$=\frac{1}{2}\begin{bmatrix}1\\i\\1\\i\end{bmatrix} \tag{20}$$

$$\Sigma_y\left(s^{(2)}+s^{(4)}\right)=\frac{1}{2}\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{bmatrix}\begin{bmatrix}-i\\1\\-i\\1\end{bmatrix} \tag{21}$$

$$=\frac{1}{2}\begin{bmatrix}-i\\1\\-i\\1\end{bmatrix} \tag{22}$$

Thus the sums u_1+v_2 and v_1+u_2 are both eigenstates of Σ_y with eigenvalue $\frac{1}{2}$. [As the u_j spinors are supposed to represent particles and the v_j antiparticles, I'm not sure what a mixture of the two is supposed to represent.]