DIRAC EQUATION: NON-RELATIVISTIC LIMIT

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.19.

The solutions to the Dirac equation consist of a 4-element column spinor and a spacetime component:

$$\left|\psi^{(1)}\right\rangle=\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c}1\\0\\\frac{p^{3}}{E+m}\\\frac{p^{1}+ip^{2}}{E+m}\end{array}\right]e^{-ipx}\equiv u_{1}e^{-ipx}\tag{1}$$

$$\left|\psi^{(2)}\right\rangle=\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c}0\\1\\\frac{p^{1}-ip^{2}}{E+m}\\-\frac{p^{3}}{E+m}\end{array}\right]e^{-ipx}\equiv u_{2}e^{-ipx}\tag{2}$$

$$\left|\psi^{(3)}\right\rangle=\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c}\frac{p^{3}}{E+m}\\\frac{p^{1}+ip^{2}}{E+m}\\1\\0\end{array}\right]e^{ipx}\equiv v_{2}e^{ipx}\tag{3}$$

$$\left|\psi^{(4)}\right\rangle=\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c}\frac{p^{1}-ip^{2}}{E+m}\\-\frac{p^{3}}{E+m}\\0\\1\end{array}\right]e^{ipx}\equiv v_{1}e^{ipx}\tag{4}$$

In the non-relativistic limit, the relative velocity of the particle satisfies $v\ll1$, which means that the momentum components all satisfy $p^{j}\ll E\approx m$. Thus $\sqrt{\frac{E+m}{2m}}\approx1$ and the solutions reduce to

$$\left|\psi^{(1)}\right\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-ipx} \tag{5}$$

$$\left|\psi^{(2)}\right\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-ipx} \tag{6}$$

$$\left|\psi^{(3)}\right\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{ipx} \tag{7}$$

$$\left|\psi^{(4)}\right\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{ipx} \tag{8}$$

Comparing this with the free particle solutions to the non-relativistic Schrödinger equation, we see that the $e^{\pm ipx} = e^{\pm Et}e^{\mp \mathbf{p}\cdot\mathbf{x}}$ factor is just what we'd get in that case. The first two components of the spinors in $\left|\psi^{(1)}\right\rangle$ and $\left|\psi^{(2)}\right\rangle$ also correspond to the eigenstates in the non-relativistic spin $\frac{1}{2}$ theory.

In the 4-d case, we can operate on these solutions with the 4-d spin operator Σ_z to get

$$\Sigma_z\left|\psi^{(1)}\right\rangle = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-ipx} \tag{9}$$

$$= \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-ipx} \tag{10}$$

$$= \frac{1}{2}\left|\psi^{(1)}\right\rangle \tag{11}$$

$$\Sigma_z\left|\psi^{(2)}\right\rangle = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-ipx} \tag{12}$$

$$= -\frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-ipx} \tag{13}$$

$$= -\frac{1}{2}\left|\psi^{(2)}\right\rangle \tag{14}$$

Similarly, we get

$$\Sigma_z\left|\psi^{(3)}\right\rangle = \frac{1}{2}\left|\psi^{(3)}\right\rangle \tag{15}$$

$$\Sigma_z\left|\psi^{(4)}\right\rangle = -\frac{1}{2}\left|\psi^{(4)}\right\rangle \tag{16}$$

These are the same results that we get by applying the Pauli spin matrices to the 2-d spin space spinors in the non-relativistic theory.