HELICITY OPERATOR IN THE DIRAC EQUATION

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.20.

In the Dirac equation, the spin of a particle is described using the spin operator

$$\Sigma_i = \frac{1}{2}\left[\begin{array}{cc} \sigma_i & 0 \\ 0 & \sigma_i \end{array}\right];\ i = x, y, z \tag{1}$$

(using natural units where $\hbar = 1$) and where the Pauli matrices are

$$\sigma_x = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \tag{2}$$

$$\sigma_y = \left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right] \tag{3}$$

$$\sigma_z = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right] \tag{4}$$

and the 0 components in 1 are 2×2 zero matrices. Spin can be oriented in any direction for particles travelling at a velocity $v < 1$, although at the speed of light ($v = 1$), the spin is always aligned with the velocity due to length contraction effects. The relation between the directions of spin and the particle's velocity is given by the *helicity*. If the 3-momentum $\mathbf{p}$ and spin both point in the same direction, the helicity has its maximum value (a positive quantity), while if they point in opposite directions, the helicity has its maximum negative value. If $\mathbf{p}$ and spin are at right angles, the helicity is zero.

These relations suggest that a helicity operator can be defined in terms of the scalar product of Σ and $\mathbf{p}$. The helicity operator is

$$\Sigma_{\mathbf{p}} \equiv \Sigma\cdot\frac{\mathbf{p}}{|\mathbf{p}|} \tag{5}$$

$$= \Sigma_1\frac{p^1}{|\mathbf{p}|} + \Sigma_2\frac{p^2}{|\mathbf{p}|} + \Sigma_3\frac{p^3}{|\mathbf{p}|} \tag{6}$$

Since the Σ_i are each a 4×4 matrix, the helicity $\Sigma_{\mathbf{p}}$ is also a 4×4 matrix, but with reference to 3-d space, it is a scalar matrix.

The solutions to the Dirac equation consist of a 4-element column spinor and a spacetime component:

$$\left|\psi^{(1)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{bmatrix} e^{-ipx} \equiv u_1 e^{-ipx} \tag{7}$$

$$\left|\psi^{(2)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \end{bmatrix} e^{-ipx} \equiv u_2 e^{-ipx} \tag{8}$$

$$\left|\psi^{(3)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \\ 1 \\ 0 \end{bmatrix} e^{ipx} \equiv v_2 e^{ipx} \tag{9}$$

$$\left|\psi^{(4)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \\ 0 \\ 1 \end{bmatrix} e^{ipx} \equiv v_1 e^{ipx} \tag{10}$$

For a particle moving in the z direction, $p^3>0$, $p^1=p^2=0$ and we've seen that the states $\left|\psi^{(1)}\right\rangle$ and $\left|\psi^{(3)}\right\rangle$ are eigenstates of Σ_3 with eigenvalue $\frac{1}{2}$. In this case, $p^3/|\mathbf{p}|=1$ so from 6

$$\Sigma_{\mathbf{p}}\left|\psi^{(1)}\right\rangle = \frac{1}{2}\left|\psi^{(1)}\right\rangle \tag{11}$$

Thus for a particle whose spin and velocity both point in the same direction, the maximum helicity value of $\frac{1}{2}$ is obtained. Similarly, if $p^3<0$, $p^1=p^2=0$ and $p^3/|\mathbf{p}|=-1$ so

$$\Sigma_{\mathbf{p}}\left|\psi^{(1)}\right\rangle = -\frac{1}{2}\left|\psi^{(1)}\right\rangle \tag{12}$$

With velocity and spin pointing in opposite directions, the maximum negative value of $-\frac{1}{2}$ is obtained for the helicity.

Now suppose we have a particle in state $\left|\psi^{(2)}\right\rangle$ and that $p^1 \neq 0$, $p^2 = p^3 = 0$. This is not a helicity eigenstate, since $\left|\psi^{(2)}\right\rangle$ is an eigenstate of Σ_3 with eigenvalue $-\frac{1}{2}$, so the spin is in the $-z$ direction while the velocity is in the x direction, so the spin is not parallel to the velocity.

Mathematically, in order for $\left|\psi^{(2)}\right\rangle$ to be a helicity eigenstate, the spinor in $\left|\psi^{(2)}\right\rangle$ would have to be an eigenstate of Σ_1, which is not true, since

$$\Sigma_1\left|\psi^{(2)}\right\rangle = \frac{1}{2}\left[\begin{array}{cccc} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{array}\right]\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c} 0\\ 1\\ \frac{p^1}{E+m}\\ 0 \end{array}\right]e^{-ipx} \tag{13}$$

$$= \frac{1}{2}\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c} 1\\ 0\\ 0\\ \frac{p^1}{E+m} \end{array}\right]e^{-ipx} \tag{14}$$

If $p^3 \neq 0$, $p^1 = p^2 = 0$, however, we *would* expect $\left|\psi^{(2)}\right\rangle$ to be a helicity eigenstate since the spin and velocity are parallel in this case. Since $\left|\psi^{(2)}\right\rangle$ is an eigenstate of Σ_3 with eigenvalue $-\frac{1}{2}$, we have

$$\Sigma_{\mathbf{p}}\left|\psi^{(1)}\right\rangle = \pm\frac{1}{2}\left|\psi^{(1)}\right\rangle \tag{15}$$

with the + corresponding to $p^3 > 0$ and the $-$ to $p^3 < 0$.

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